学位論文

Quantum field theoretical approach to relativistic hydrodynamics from local Gibbs ensemble

(局所ギブス分布に基づく相対論的流体力学に対する
場の量子論的アプローチ)

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Quantum field theoretical approach to relativistic hydrodynamics from local Gibbs ensemble

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Abstract

Relativistic hydrodynamics is a low-energy effective theory which universally describes macroscopic behaviors of relativistic many-body systems. Its application covers the broad branches of physics such as high-energy physics, astrophysics, and condensed matter physics. Nevertheless, its derivation from underlying microscopic theories, especially from quantum field theories, has not been clearly clarified. Furthermore, it is recently pointed out that novel transport phenomena such as the chiral magnetic effect, which originates from the quantum anomaly in the underlying quantum field theories, take place in the macroscopic hydrodynamic regime.

In this thesis, we derive relativistic hydrodynamics from quantum field theories on the basis of the recent development of nonequilibrium statistical mechanics. In order to derive the hydrodynamic equations we introduce an assumption that the density operator is given by a local Gibbs distribution at initial time, and decompose the energy-momentum tensor and charge current into nondissipative and dissipative parts. This leads to a generalization of the Gibbs ensemble method canonically employed in equilibrium statistical mechanics. Our formalism is also applicable to the situation in the presence of the quantum anomaly, and we can describe the anomaly-induced tranport phenomena.

As a basic tool of our formalism, we first develop a path integral of the thermodynamic potential for locally thermalized systems. We show microscopically that the thermodynamic pontential, which is shown to be the generating functional of systems in local thermal equilibrium, is written in terms of the quantum field theory in the curved spacetime with one imaginary-time direction. The structure of this thermally emergent curved spacetime is determined by hydrodynamic variables such as the local temperature, and fluid-four velocity, and possesses notable symmetry properties: Kaluza-Klein gauge symmetry, spatial diffeomorphism symmetry, and gauge symmetry. With the help of the symmetry argument, we can construct the nondissipative part of the hydrodynamic equations including the anomaly-induced transport phenomena. By the use of the perturbative calculation, we evaluate the anomalous transport coefficients at one-loop level. Furthermore, we also construct a solid basis to study dissipative corrections to hydrodynamic equations. In particular, by performing the derivative expansion, together with the result on nondissipative part of the constitutive relations, we derive the first-order dissipative hydrodynamic equations, that is, the relativistic Navier-Stokes equation. Our formalism also provides the quantum field theoretical expression of the Green-Kubo formulas for transport coefficients.

The formulation and whole works presented from Chapter 2 to Chapter 4 are based on our original work in collaboration with Yoshimasa Hidaka (RIKEN). The work presented in Chapter 3 is in collaboration with Yoshimasa Hidaka (RIKEN), Tomoya Hayata (RIKEN), and Toshifumi Noumi (Hong Kong University of Science and Technology) [1].

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Chapter 1

Introduction

Hydrodynamics has a long history of theoretical study and its application to our real life. Its prototype, hydrostatics, has already been started in seventeenth century by Pascal's work on the so-called Pascal's law [2]. Newton also worked on hydrostatics in his monumental "Philosophiæ Naturalis Principia Mathematica" [3]. While their works were restricted to static phenomena of fluid, and were indeed not hydro*dynamics* but hydro*statics*, Daniel Bernoulli opened a new chapter of systematic study on hydrodynamics in his "Hydrodynamica, sive de viribus et motibus fluidorum commentarii" in 1738 [4]. He addressed several issues not only on a motion of fluid itself but also a first attempt to make a use of kinetic theory of gases to derive the Boyle's law. After seminal works by Clairaut and d'Alembert [5, 6, 7], Euler and Lagrange have laid the foundation of hydrodynamics[8, 9, 10]. Their framework of hydrodynamics, nevertheless, still remained irrelevant to the mundane problems such as pipe flows since they cannot cover the effects of viscosity. In the early nineteenth century, French physicists such as Navier, Cauchy, Saint-Venant, and Stokes worked on this problem and finally obtain the basic equation of hydrodynamics applicable to realistic situations, which is now known as the so-called Navier-Stokes equation[1].After the derivation of the Navier-Stokes equation almost 200 years ago, there still exist rich unsolved problems related to hydrodynamics ranging from mathematical one such as the "Navier-Stokes existence and smoothness" in Millennium Prize Problems by Clay Mathematics Institute to technological or engineering ones like the aircraft design.

Restricting ourselves to problems in physics, we often encounter hydrodynamics on the front line of research in various fields. This is because hydrodynamics gives a universal description of the macroscopic behaviors of many-body systems [12], focusing only on the spacetime evolution of densities of conserved charges such as energy, momentum, and electric charge. In fact, the framework of hydrodynamics itself does not depend on microscopic details of systems such as the nature of particles and their interactions, and its application covers various branches of physics from condensed matter to high-energy physics. Among them is illuminating the recent success of relativistic hydrodynamics in describing the spacetime evolution of the quark-gluon

[1] There is close investigation of priority to the Navier-Stokes equation in Ref. [11].

plasma (QGP) created in ultra-relativistic heavy-ion collision experiments [13, 14, 15, 16, 17, 18, 19]. In parallel with hydrodynamic modeling of the QGP, relativistic hydrodynamics itself has attracted much attention. The first-order relativistic hydrodynamic equations, that is, the relativistic version of the Navier-Stokes equations, which suffer from the violation of causality, have been derived long ago by Eckart [20] and by Landau and Lifshitz [12]. The second-order equations, which resolve the causality problem by introducing a finite relaxation time, were derived first by Müller [21] and also by Israel and Stewart [22]. After the aforementioned success of relativistic hydrodynamics in describing the QGP, a lot of works concerning the derivation of the relativistic hydrodynamic equations have been progressively carried out. For example, the relativistic hydrodynamic equations are formulated based on the kinetic theory [23, 24, 25, 26, 27, 28, 29, 30, 31, 32], the fluid/gravity correspondence [33, 34, 35, 36], the phenomenological extension of nonequilibrium thermodynamics [37, 38], and the projection operator method [39, 40]. Also, a novel method has recently been developed in which the generating functional for nondissipative hydrodynamics in equilibrium is constructed only by imposing symmetry and scaling properties of systems [41, 42].

Despite the progresses mentioned above, the foundation of hydrodynamics based on underlying microscpobic theories, especially quantum field theories, has not fully understood yet. In fact, it has been recently pointed out that a set of novel transport phenomena, which originate from quantum anomalies of underlying quantum field theories, take place in a medium composed of chiral fermions. They are called the anomaly-induced transport phenomena. One of such examples is the chiral magnetic effect which represents the existence of the electric current along the magnetic field. The chiral magnetic effect only arises when a system is under parity-violating environments expected to be realized in the QGP in heavy-ion collisions, hot and dense neutrino gases in the supernovae explosion, and also a Weyl semimetal in condensed matter. Although the parity-violating transport itself was already pointed out more than 30 years ago [43, 44] it is clearly recognized only recently that they appear even in the macroscopic hydrodynamic regime as a result of the quantum anomaly. The anomaly-induced transport phenomena are distinct from the usual transport phenomena in hydrodynamics such as the conducting current, and have not been fully understood from the point of the view of nonequilibrium statistical mechanics.

The aim of this thesis is to derive the relativistic hydrodynamic equations on the basis of the underlying microscopic theories, that is, quantum field theories. To derive the hydrodynamic equations, we introduce the local Gibbs distribution at initial time, which is a generalization of the Gibbs ensemble methods [45] usually applied in the equilibrium statistical mechanics [46]. This is based on the recent development of nonequilibrium statistical mechanics [47], which is quite similar with methods using nonequilibrium statistical ensembles [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. However, we first performed the path-integral analysis of the thermodynamic potential for locally thermalized systems in detail, and formulated quantum field theories under local thermal equilibrium. This provides a first microscopic justification of the generating functional method [41, 42] for nondissipative hydrodynamics, and enables us to justify a generalized argument by Luttinger [58], in which the spatial distribution of the temperature is interpreted as an auxiliary external gravitational potential. This lays out a solid basis to describe the anomaly-induced transport phenomena based on the underlying quantum field theory. We also study the dissipative corrections to relativistic hydrodynamic equations by using our method and show that our formulation provides a solid basis not only to derive the first-order equations but also to proceed to higer orders. Moreover, we consider the system composed of the chiral fermion, and derive the anomaly-induced transport as a first-order nondissipative correction to the constitutive relations.

This chapter is organized as follows: In Sec. 1.1, we first reconsider the basis of relativistic hydrodynamics in detail. Then, we review different ways of constructing hydrodynamics: the phenomenological derivation, the derivation based on the kinetic theory, and the method based on the hydrostatic partition function. In Sec. 1.2, we introduce the anomaly-induced transport phenomena, whose derivation is one of the main topic in Chapter 4. We also show some examples where such transport phenomena take place. In Sec. 1.3, the outline of this thesis is shown.

1.1 Relativistic hydrodynamics

In this section, we present a basis of hydrodynamics. In Sec. 1.1.1, after explaining the fundamental assumption to apply hydrodynamics, we list a set of relavant variables which live in the hydrodynamic regime. In Sec. 1.1.2, we set out the building blocks for hydrodynamics: the conservation laws for the macroscopic variables, the constitutive relations, and the set of physical properties of systems. In 1.1.3, we briefly review some conventional derivations of relativistic hydrodynamic equations.

1.1.1 What is hydrodynamics?

Applicable condition for hydrodynamics

Hydrodynamics gives a systematic and powerful way to describe the spacetime evolution of many-body systems, or systems composed of quantum fields. However, it is not always applicable in general nonequilibrium situations. Here we demonstrate the conditions under which hydrodynamics becomes a proper tool to describe the real-time evolution of systems.

Given a certain Lagrangian (or Hamiltonian), we have intrinsic microscopic scales such as the mean free path and mean free time which are determined by the nature of particles and interactions between them. Let ℓ_{micro} denote such a microscopic scale. Under some circumstances like ones in which local thermodynamics is even approximately applicable, we also have macroscopic length scale ℓ_{macro}. Let this macroscopic scale ℓ_{macro} characterize the minimum scale of the macroscopic behaviors of systems such as a scale of steepest temperature gradient of the systems. Then, the fundamental assumption to apply hydrodynamics is that there exists a scale separation between ℓ_{micro} and ℓ_{macro}. In other words, the existence of a cut-off scale Λ satisfying

$$\ell_{\text{micro}} \ll \Lambda \ll \ell_{\text{macro}}, \tag{1.1}$$

is necessary to describe systems with the help of hydrodynamics.

The presence of this hierarchy allows us to describe systems by the use of coarse graining, i.e. the average over the scale shorter than the cut-off scale Λ. Since this cut-off scale is large compared to the microscopic one, the average values of conserved charge densities are expected to take typical values. Furthermore, if we focus on macroscopic behaviors of the averaged conserved charge densities, they are independent of the cut-off scale because the cut-off scale is sufficiently small compared to our macroscopic scale. Then, introducing a scale separation parameter $\epsilon \equiv \ell_{\text{micro}}/\ell_{\text{macro}}$, we consider the situation that the systems are characterized by a set of parameters $\lambda^a(x)$ which has sufficiently smooth coordinate dependence: $\lambda^a(x) = \bar{\lambda}^a(\epsilon x)$, where the functional form of $\bar{\lambda}^a$ does not depend on ϵ. Therefore, derivatives of relevant parameters gives higher-order contribution, and we can employ the derivative expansion for the relevant variables. This is the fundamental condition for hydrodynamics to work.

Relevant variables in hydrodynamics

From the modern point of view, hydrodynamics is regarded as a low energy effective theory which describes low-frequency and low-wavenumber behaviors of many-body systems. As is usual in order to construct an effective theory, we first have to specify a set of the relevant variables in the hydrodynamic description. In other words, we have to clarify what is a complete set of $\lambda^a(x)$ in the hydrodynamic regime.

Through the procedure of coarse graining, or averaging within a cut-off scale, a great number of degrees of freedom is reduced, and relevant variables which appear in the hydrodynamic description of systems are strongly restricted. In fact, the only relevant variables which live in the hydrodynamic regime are as listed below[2]:

- Conserved quantities, e.g. energy, momentum, and conserved charge (like electric charge)
- Nambu-Goldstone mode associated with spontaneous symmetry breaking
- Electromagnetic field ($U(1)$ gauge field), especially magnetic field[3]

The reason why these still remain is easily understood by focusing on the dispersion relations. If we have the massless dispersion relation, or $\omega \propto k^\alpha$ with positive α, the associated modes

[2] We have also fluctuations of the order parameters, or the critical fluctuation, in the vicinity of a critical point, where a second-order phase transition takes place.

[3] Similar to the electromagnetic field, we also have the gravitational field as an additional relevant variable when we consider the dynamics of the spacetime itself.

survive in the macroscopic scale, in which the mode becomes arbitrarily slow ($\omega \to 0$) when we observe in the enough macroscopic scale ($k \to 0$). Then, all the modes listed above have massless dispersion relations, and they survive. In fact since we have the conservation laws, the conserved quantities have the massless dispersion relation. Also the Nambu-Goldstone mode has the massless one due to the occurrence of a flat direction of the effective potential. While electric fields in the charged plasma is screened, the magnetic field is not, and it remains as one of the massless modes.

We have clarified the relevant variables in the hydrodynamic regime, and we next explain what will be resulting hydrodynamic equations. If we only have the conserved quantities, it leads to the usual hydrodynamic equations such as the Euler equation for a perfect fluid, or the Navier-Stokes equation depending on the order of the derivative expansion. In the coexistence of the conserved quantities and Nambu-Goldstone modes associated with the spontaneous symmetry breaking, the usual hydrodynamic equations are modified. The modified equations including the Nambu-Goldstone mode results in the two-fluid hydrodynamic equations, which is well known in the case of the superfluid ^{4}He. Finally if we have the dynamical electromagnetic fields, the resulting equation is the magneto-hydrodynamic equations.

In addition, it has been recently pointed out that the existence of the quantum anomaly strongly affects hydrodynamics. It actually induces novel transport phenomena while it does not introduce the new degrees of freedoms. This novel tranport phenomena is introduced in Sec. 1.2, and is one of the main subjects of this thesis. Hydrodynamics including the quantum anomaly and anomaly-induced transport is called anomalous hydrodynamics.

1.1.2 Building blocks of hydrodynamics

Conservation law for macroscopic variables

Here, we consider only conserved quantities as hydrodynamic variables. Then the basic building blocks for hydrodynamics are the conservation laws for the energy-momentum and conserved charge such as electric charge:

$$\nabla_\mu \hat{T}^{\mu\nu}(x) = 0, \tag{1.2}$$

$$\nabla_\mu \hat{J}^{\mu}(x) = 0, \tag{1.3}$$

where $\hat{T}^{\mu\nu}$ and $\hat{J}^\mu$ ($\mu, \nu = 0, 1, 2, \cdots, d-1$) are the energy-momentum tensor and conserved current operator such as vector current, respectively. Here considering a general case, we introduce the covariant derivative ∇_μ. These are equations for the operator composed of quantum fields without coarse graining. By taking the average over an appropriate density operator $\hat{\rho}_0$, which represents the average within a cut-off scale discussed in the previous subsection, we obtain the conservation laws for the expectation values of the energy-momentum tensor and the conserved current

$$\nabla_\mu \langle \hat{T}^{\mu\nu}(x) \rangle = 0, \tag{1.4}$$
$$\nabla_\mu \langle \hat{J}^{\mu}(x) \rangle = 0, \tag{1.5}$$

where $\langle \hat{\mathcal{O}}(x) \rangle \equiv \mathrm{Tr}\, \hat{\rho}_0 \hat{\mathcal{O}}(x)$. This set of the conservation laws is the basic equation of relativistic hydrodynamics. Here relevant dynamical variables are conserved charge densties such as $\langle \hat{T}^{0\nu}(x) \rangle$ and $\langle \hat{J}^{0}(x) \rangle$.

Constitutive relation

We cannot solve the conservation laws (1.4) and (1.5) without relating $\langle \hat{T}^{i\nu}(x) \rangle$ and $\langle \hat{J}^{i}(x) \rangle$ $(i = 1, 2, \cdots, d-1)$ to the conserved charge densities $\langle \hat{T}^{0\nu}(x) \rangle$ and $\langle \hat{J}^{0}(x) \rangle$, or their conjugate variables $\beta^\mu(x)$ and $\nu(x)$[4]. Then, the second building blocks for hydrodynamics are the constitutive relations

$$\langle \hat{T}^{\mu\nu}(x) \rangle = T^{\mu\nu}[T^{0\nu}, J^0] = T^{\mu\nu}[\beta^\mu, \nu], \tag{1.6}$$
$$\langle \hat{J}^{\mu}(x) \rangle = J^{\mu}[T^{0\nu}, J^0] = J^{\mu}[\beta^\mu, \nu]. \tag{1.7}$$

Once we obtain the constitutive relations (1.6) and (1.7), together with the conservation laws, we have the complete set of hydrodynamic equations. This format of the conservation laws and constitutive relations is universal and independent of what kind of microscopic constituents we are considering.

Equation of state and transport coefficients

Although the hydrodynamic equations give universal description of macroscopic behaviors of systems, motions of fluid itself depends on the physical properties of systems determined by the microscopic details. The microscopic feature of systems such as the nature of particles and their interactions, is summarized in a few equations: the equation of state, and transport coefficients. For example, we have to specify the equation of state

$$p = p(e, n) = p(\beta, \nu), \tag{1.8}$$

where p denotes the pressure of the fluid, which appears as the diagonal spatial component of the averaged energy-momentum tensor in the leading order. Here e and n are energy density and charge density of fluid, respectively, and β and ν are their conjugate variables.

The transport coefficients, such as the shear viscosity η, bulk viscosity ζ, and charge conductivity κ, appear in the next-to-leading order constitutive relations. Let L_i denote a set

[4] As is discussed in the subsequent chapter, these conjugate variables are indeed related to local temperature, fluid-four velocity, and local chemical potential.

of transport coefficients: $L_i \equiv \{\eta,\ \zeta,\ \kappa\}$. Then, we have to specify the values of transport coefficients

$$L_i = L_i(e, n) = L_i(\beta, \nu). \tag{1.9}$$

The funcional forms of Eqs. (1.8) and (1.9) depend on the microscopic details of systems dictated e.g. by the coupling constants of quantum fields.

1.1.3 Conventional derivations of hydrodynamics

Phenomenological derivation of hydrodynamics

Here we briefly review a phenomenological way to derive hydrodynamic equations based on local thermodynamics [12]. We first put an assumption that equilibrium thermodynamics is satisfied in the frame moving with the fluid velocity $\boldsymbol{v}$, which is called the local rest frame. Then, considering the physical meaning of a each component of the energy-momentum tensor and charge current, we can write down their expressions in the local rest frame as

$$T^{\mu\nu}_{\rm LRF}(x) = \begin{pmatrix} e(x) & 0 & 0 & 0 \\ 0 & p(x) & 0 & 0 \\ 0 & 0 & p(x) & 0 \\ 0 & 0 & 0 & p(x) \end{pmatrix}, \quad J^{\mu}_{\rm LRF}(x) = \begin{pmatrix} n(x) \\ 0 \end{pmatrix}, \tag{1.10}$$

where $e(x)$, $p(x)$, and $n(x)$ denote the energy density, pressure, and charge density of the fluid, respectively. By the use of the Lorentz transformation, we obtain the zeroth-order constitutive relation

$$T^{\mu\nu}_{(0)} = (e+p)u^\mu u^\nu + p g^{\mu\nu}, \tag{1.11}$$

$$J^{\mu}_{(0)} = n u^\mu, \tag{1.12}$$

where the four-fluid velocity $u^\mu = \gamma(x)(1, \boldsymbol{v}(x))$ with the Lorentz contraction factor $\gamma(x) \equiv 1/\sqrt{1-\boldsymbol{v}^{\,2}(x)}$. The conservation laws and the above constitutive relations result in the Euler equation for a perfect fluid.

Since we assume that local fluid variables such as $e(x)$, $p(x)$, $n(x)$ obey thermodynamics, we have first law of thermodynamics:

$$e + p = sT + \mu n, \tag{1.13}$$

$$de = T ds + \mu dn, \tag{1.14}$$

where we introduced the entropy density $s(x)$, local temperature $T(x)$, and local chemical potential $\mu(x)$ of the fluid. By using the equation of motion for the zeroth-order hydrodynamics: $\nabla_\mu T^{\mu\nu}_{(0)} = 0$, $\nabla_\mu J^{\mu}_{(0)} = 0$, together with the first law of thermodynamics, we obtain

$$\nabla_\mu s^\mu_{(0)} = 0, \quad \text{with} \quad s^\mu_{(0)} = s u^\mu, \tag{1.15}$$

where $s(x)$ is the entropy density of the fluid defined in Eq. (1.13). Evidently we have the conserved entropy current $s^\mu_{(0)}$ in the zeroth-order hydrodynamics. This is because the perfect fluid does not contain the effect of dissipation like the viscosity. The spacetime evolution of the perfect fluid is regarded as adiabatic procedures.

We, then, proceed to the first-order hydrodynamics. In order to consider the effect of dissipation, we express the constitutive relation as

$$T^{\mu\nu}_{(1)} = (e+p)u^\mu u^\nu + pg^{\mu\nu} + \delta T^{\mu\nu}_{(1)}, \tag{1.16}$$

$$J^\mu_{(1)} = nu^\mu + \delta J^\mu_{(1)}, \tag{1.17}$$

where $\delta T^{\mu\nu}_{(1)}$ and $\delta J^\mu_{(1)}$ are dissipative derivative corrections in the first-order constitutive relations, which are orthogonal to u^μ: $\delta T^{\mu\nu}_{(1)} u_\nu = \delta J^\mu_{(1)} u_\mu = 0$. Then, our problem is to write down the form of the $\delta T^{\mu\nu}_{(1)}$ and $\delta J^\mu_{(1)}$. To obtain them, we introduce a crucial assumption that there exists the entropy current $s^\mu_{(1)}$ such that

$$\nabla_\mu s^\mu_{(1)} \geq 0, \tag{1.18}$$

is satisfied. Here the entropy current is also modified and written as

$$s^\mu_{(1)} = su^\mu + \delta s^\mu_{(1)}, \tag{1.19}$$

By the use of the modified constitutive relations (1.16) and (1.17) and the modified entropy current (1.19), together with the first law of thermodynamics (1.13) and (1.14), we obtain

$$\nabla_\mu \left(su^\mu - \nu \delta J^\mu \right) = -\delta J^\mu \nabla_\mu \nu - \beta \delta T^{\mu\nu} \nabla_\mu u_\nu, \tag{1.20}$$

where we introduced $\beta \equiv 1/T$, and $\nu \equiv \beta\mu$. This equation enables us to construct $\delta T^{\mu\nu}_{(1)}$, $\delta J^\mu_{(1)}$, and $\delta s^\mu_{(1)}$ satisfying the local version of the thermodynamic second law as

$$\delta T^{\mu\nu}_{(1)} = -2\eta \nabla^{\langle\mu} u^{\nu\rangle} - \zeta (\nabla_\alpha u^\alpha) \Delta^{\mu\nu}, \tag{1.21}$$

$$\delta J^\mu_{(1)} = -\frac{\kappa}{\beta} \nabla^\mu_\perp \nu, \tag{1.22}$$

$$\delta s^\mu_{(1)} = \kappa\mu \nabla^\mu_\perp \nu, \tag{1.23}$$

with positive η, ζ, and κ, which are transport coefficients and called the shear viscosity, bulk viscosity, and charge conductivity, respectively. Here we defined $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$, which satisfies $\Delta^{\mu\nu} u_\nu = 0$, and $\nabla^\mu_\perp \equiv \Delta^{\mu\nu} \nabla_\nu$. We also introduced the angle bracket as the projection to traceless part given by

$$\nabla^{\langle\mu} u^{\nu\rangle} \equiv \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{1}{d-1} \Delta^{\mu\nu} \Delta^{\alpha\beta} \nabla_\alpha u_\beta, \tag{1.24}$$

with spacetime dimension d. Substituting these constitutive relations in Eq. (1.20), we obtain $\nabla_\mu s^\mu_{(1)} \geq 0$. Therefore, we obtain the first-order constitutive relations given by

$$T^{\mu\nu}_{(1)} = (e+p)u^\mu u^\nu + pg^{\mu\nu} - 2\eta \nabla^{\langle\mu} u^{\nu\rangle} - \zeta (\nabla_\alpha u^\alpha) \Delta^{\mu\nu}, \tag{1.25}$$

$$J^\mu_{(1)} = nu^\mu - \frac{\kappa}{\beta} \nabla^\mu_\perp \nu. \tag{1.26}$$

These constitutive relations result in the relativistic version of the Navier-Stokes equation. This phenomenological derivation is quite simple and extendable in the presence of the quantum anomalies [59]. However, it should be emphasized that we do not have the microscopic formulas to determine the transport coefficients, and, thus, they are dealt with phenomenological parameters to be determined from experiments. This is because we adopted the *ad hoc* assumption on the phenomenological use of the local second law of thermodynamics, and stayed away statistical mechanics.

Derivation from the Boltzmann equation

Another way to construct the constitutive relations is to use the kinetic theory, or the relativistic Boltzmann equation [60, 61, 62, 63, 64], which is derived from weak-coupling quantum field theories,

$$p^\mu \partial_\mu f(x,p) = C[f](x,p), \tag{1.27}$$

where $f(x,p)$ denotes the one particle distribution function for the momentum p^μ at the position x^μ. Here the right-hand side represents the collision term given by

$$\begin{aligned} C[f](x,p) = \frac{1}{2}\int \frac{d^3\boldsymbol{p}_1}{(2\pi)^3 E_{p_1}} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3 E_{p_2}} \frac{d^3\boldsymbol{p}_3}{(2\pi)^3 E_{p_3}} \mathcal{W}(p,p_1;p_2,p_3) \\ \times \left[(1+af_p)(1+af_{p_1}) f_{p_2} f_{p_3} - f_p f_{p_1}(1+af_{p_2})(1+af_{p_3})\right], \end{aligned} \tag{1.28}$$

where $\mathcal{W}(p,p_1,;p_2,p_3)$ is the transition matrix, and f_p is the short-hand notation of $f(x,p)$: $f_p = f(x,p)$. The transition matrix usually satisfies the symmetry properties: $\mathcal{W}(p,p_1;p_2,p_3) = \mathcal{W}(p_2,p_3;p,p_1) = \mathcal{W}(p_1,p;p_3,p_2) = \mathcal{W}(p_3,p_2;p_1,p)$. Here a represents statistics of particles: $a=+1$ for bosons, $a=-1$ for fermions, and $a=0$ for classical particles. With the help of the one particle distribution function, we can define the energy-momentum tensor $T^{\mu\nu}(x)$ and charge current $J^\mu(x)$ as

$$T^{\mu\nu}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_p} p^\mu p^\nu f(x,p), \tag{1.29}$$

$$J^\mu(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_p} p^\mu f(x,p). \tag{1.30}$$

Using a symmetric properties of the transition matrix and the Boltzmann equation, we can show the conservation laws:

$$\begin{aligned} \partial_\mu T^{\mu\nu}(x) &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_p} p^\mu p^\nu \partial_\mu f(x,p) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_p} p^\nu C[f] \\ &= 0, \end{aligned} \tag{1.31}$$

$$\begin{aligned} \partial_\mu J^\mu(x) &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_p} p^\mu \partial_\mu f(x,p) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_p} C[f] \\ &= 0. \end{aligned} \tag{1.32}$$

Then, our problem is to express these conserved charge currents in terms of thermodynamic variables. In order to do so, we perform the tensor decomposition

$$T^{\mu\nu} = eU^\mu U^\nu + P\Delta^{\mu\nu} + q^\mu U^\nu + q^\nu U^\mu + \pi^{\mu\nu}, \tag{1.33}$$

$$J^\mu = nU^\mu + \nu^\mu, \tag{1.34}$$

where U^μ is arbitrary vector at this stage, $\Delta^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$ the projection operator orthogonal to U^μ, q^μ and ν^μ vectors orthogonal to U^μ, and $\pi^{\mu\nu}$ the symmetric traceless part of the energy-momentum tensor also orthogonal to U^μ.

Let us consider systems slightly deviating from thermal equilibrium and derive the constitutive relations. In that case, we may expand the distribution function around local thermal equilibrium

$$f(x,p) = f^{(0)}(x,p) + \delta f(x,p), \tag{1.35}$$

where $f^{(0)}(x,p)$ denotes the local equilibrium distribution function given by

$$f^{(0)}(x,p) = \frac{1}{e^{-\beta^\mu(x)p_\mu + \nu(x)} - a}, \tag{1.36}$$

where $\beta^\mu \equiv \beta u^\mu$, and $\nu = \beta\mu$ with local inverse temperature β, fluid-four velocity u^μ, and local chemical potential μ. Here, to decompose the thermal equilibrium part $f^{(0)}(x,p)$ and the deviation $\delta f(x,p)$, we impose the so-called matching condition

$$\delta T^{\mu\nu} u_\nu = 0, \quad \text{with} \quad \delta T^{\mu\nu}(x) \equiv \int \frac{d^3p}{(2\pi)^3 E_p} p^\mu p^\nu \delta f(x,p), \tag{1.37}$$

$$\delta J^\mu u_\nu = 0, \quad \text{with} \quad \delta J^\mu(x) \equiv \int \frac{d^3p}{(2\pi)^3 E_p} p^\mu \delta f(x,p). \tag{1.38}$$

This condition enables us to apply local thermodynamics in the fluid-comoving frame, and to relate the conserved charge densities to the local thermodynamic parameters such as $\beta^\mu(x)$ and $\nu(x)$. After taking this matching condition, we choose $U^\mu(x) = u^\mu(x)$, and obtain

$$T^{\mu\nu} = eu^\mu u^\nu + (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{1.39}$$

$$J^\mu = nu^\mu + \nu^\mu, \tag{1.40}$$

where we write $P = p + \Pi$ with the thermodynamic pressure p determined from $f_p^{(0)}$, and the deviation as Π. We note that δf_p in Eq. (1.35) represents the deviation from local thermal equilibrium, and gives the dissipative corrections. Our problem is now to write down Π, $\pi^{\mu\nu}$ and ν^μ in terms of hydrodynamic variables like u^μ, β, μ, and their derivatives. This program can be successfully accomplished by treating δf_p in a perturbative way in various methods such as the Chapman-Enskog method [65, 66, 67], the relaxation time approximation method [68, 32], the Grad's fourteen moment method [69, 70, 22, 23, 27, 29, 31], and the renormalization group method [71, 24, 25, 72, 73].

Here we briefly show the derivation based on the Chapman-Enskog method [60, 65, 66, 67]. First of all, decomposing the derivative as

$$\partial_\mu = \delta^\nu_\mu \partial_\nu = -u_\mu D + \partial_{\perp\mu}, \quad \text{with} \quad D \equiv u^\mu \partial_\mu, \quad \partial_{\perp\mu} \equiv \Delta^{\mu\nu}\partial_\nu, \tag{1.41}$$

we rewrite the Boltzmann equation as

$$-Df_p = \frac{1}{p\cdot u} C[f]_p - \epsilon \frac{1}{p\cdot u} p^\mu \partial_{\perp\mu} f_p, \tag{1.42}$$

where ϵ is a bookkeeping parameter representing the scale separation corresponding to the Knudsen number: $\epsilon = \ell_{\text{micro}}/\ell_{\text{macro}}$. Then, we expand the distribution function as

$$f_p = f_p^{(0)} + \epsilon f_p^{(1)} + \mathcal{O}(\epsilon^2), \tag{1.43}$$

$$Df_p = \epsilon (Df_p)^{(1)} + \mathcal{O}(\epsilon^2), \tag{1.44}$$

where we used the fact that the proper time derivative of the one particle distribution function vanishes in the zeroth-order: $(Df_p)^{(0)} = 0$. After linearizing the Boltzmann equation on top of $f_p^{(0)}$, the first-order equation with respect to ϵ results in

$$-(p\cdot u)(Df_p)^{(1)} + p^\mu \partial_{\perp\mu} f_p^{(0)} = f_p^{(0)}(1 + a f_p^{(0)}) L[\phi^{(1)}]_p, \tag{1.45}$$

where we defined $f_p^{(1)} \equiv f_p^{(0)}(1 + a f_p^{(0)})\phi_p^{(1)}$, and introduced the linearized collision operator $L[\phi^{(1)}]$ as

$$\begin{aligned} L[\phi](x,p) = -\frac{1}{2(1 + a f_p^{(0)})} \int & \frac{d^3\boldsymbol{p}_1}{(2\pi)^3 E_{p_1}} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3 E_{p_2}} \frac{d^3\boldsymbol{p}_3}{(2\pi)^3 E_{p_3}} \mathcal{W}(p, p_1; p_2, p_3) \\ & \times f_{p_1}^{(0)}(1 + a f_{p_2}^{(0)})(1 + a f_{p_3}^{(0)}) \left[\phi_p + \phi_{p_1} - \phi_{p_2} - \phi_{p_3}\right]. \end{aligned} \tag{1.46}$$

The first term in Eq. (1.45) can be calculated as

$$(Df_p)^{(1)} = \frac{\partial f_p^{(0)}}{\partial \beta} D\beta + \frac{\partial f_p^{(0)}}{\partial \nu} D\nu + \frac{\partial f_p^{(0)}}{\partial u^\mu} Du^\mu. \tag{1.47}$$

Then, with the help of the zeroth-order hydrodynamic equation, we have

$$D\beta = \beta \left(\frac{\partial p}{\partial e}\right)_n \theta, \tag{1.48}$$

$$D\nu = -\beta \left(\frac{\partial p}{\partial n}\right)_e \theta, \tag{1.49}$$

$$Du^\mu = T\partial_{\perp\mu}\beta - \frac{nT}{h}\partial_{\perp\mu}\nu, \tag{1.50}$$

where $\theta \equiv \partial_\mu u^\mu$, and $h \equiv e + p$ denotes the enthalpy density. Therefore, we obtain the first-order linearized Boltzmann equation

$$-\beta\theta\Pi_p - \beta\sigma_{\mu\nu}\pi_p^{\mu\nu} - (\partial_{\perp\mu}\nu) J_p^\mu = L[\phi^{(1)}]_p, \tag{1.51}$$

where we defined the projected tensor decomposition of the energy-momentum and current as

$$\Pi_p \equiv p^\mu p^\nu \left(\frac{\Delta_{\mu\nu}}{d-1} - p_e u_\mu u_\nu \right) - p\cdot u p_n \tag{1.52}$$

$$\pi_p^{\mu\nu} \equiv p^{\langle\mu} p^{\nu\rangle}, \tag{1.53}$$

$$J_p^\mu \equiv p_\perp^\mu \left(1 - p\cdot u \frac{n}{h} \right). \tag{1.54}$$

Here we introduced $p_e \equiv (\partial p/\partial e)_n$, $p_n \equiv (\partial p/\partial n)_e$. From Eq. (1.51), we can express $\phi_p^{(1)}$ as

$$\phi_p^{(1)} = -\beta\theta L^{-1}[\Pi_p] - \beta\sigma_{\mu\nu} L^{-1}[\pi_p^{\mu\nu}] - \partial_{\perp\mu}\nu L^{-1}[J_p^\mu]. \tag{1.55}$$

This expression allows us to write down the constitutive relations by using $f_p^{(1)} = f_p^{(0)}(1 + af_p^{(0)})\phi_p^{(1)}$, and they are again given by the same expressions as Eqs. (1.25) and (1.26). In addition to the constitutive relations, we have the expressions for the transport coefficients as

$$\zeta = \beta\langle \Pi_p, L^{-1}[\Pi_p]\rangle, \tag{1.56}$$

$$\eta = \frac{\beta}{(d+1)(d-2)} \langle \pi_p^{\mu\nu}, L^{-1}[\pi_p^{\rho\sigma}]\rangle \Delta_{\mu\rho}\Delta_{\nu\sigma}, \tag{1.57}$$

$$\kappa = \frac{\beta}{d-1} \langle J_p^\mu, L^{-1}[J_p^\nu]\rangle \Delta_{\mu\nu}, \tag{1.58}$$

where we defined the inner product as

$$\langle A_p, B_p\rangle \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_p} f_p^{(0)}(1 + af_p^{(0)}) A_p B_p. \tag{1.59}$$

Therefore, if we specify the dynamics of the system, we can, in principle, calculate the transport coefficient with the help of these equations, which give the kinetic expression of the Green-Kubo formulas [74, 75, 76]. This gives one derivation of the hydrodynamic equation on the basis of the Boltzmann equation. Nevertheless, it is worth while to point out a weakness of the derivation based on the kinetic theory. While the Boltzmann equation gives us a strong tool to derive the constitutive relations, and also actual values of transport coefficients through the Green-Kubo formulas, it is only valid in the weak-coupling regime. On the other hand, hydrodynamics still works in the strong-coupling regime, and, thus, the derivation should not be restricted to the kinetic theory. Moreover, as we will introduce in the next section, the anomaly-induced transport phenomena is not described in a straightforward manner in the conventional kinetic theory demonstrated here. Although a new way to describe them in the framework of the kinetic theory has been recently developed, the form of the collision term has been still elusive [77, 78, 79, 80, 81, 82, 83, 84]. Therefore, there is a compelling need for the derivation of hydrodynamics without relying on the kinetic theory.

Derivation using hydrostatic partition function method

A novel method has recently been developed in which the equilibrium-generating functional for nondissipative hydrodynamics is constructed only by imposing the symmetry and scaling properties of systems [41, 42]. In these works, relativistic quantum field theories on a general manifold with a timelike killing vector, whose line element ds^2 is parametrized by the Kaluza-Klein form, is considered under the presence of a time independent background $U(1)$ gauge connection $\mathcal{A}$:

$$ds^2 = -e^{2\sigma(\boldsymbol{x})}\left(dt + a_i(\boldsymbol{x})dx^2\right)^2 + g_{ij}(\boldsymbol{x})dx^i dx^j, \tag{1.60}$$

$$\mathcal{A} = \mathcal{A}_0(\boldsymbol{x})dx^0 + \mathcal{A}_i(\boldsymbol{x})dx^i, \tag{1.61}$$

where σ, a_i , g_{ij} are smooth functions only dependent on spatial coordinate $\boldsymbol{x}$. Only based on general symmetry and scaling grounds, the possible form of the generating functional reads

$$\log Z = \int d^{d-1}x\sqrt{g}\frac{e^{\sigma}}{T_0}p(T_0 e^{-\sigma}, e^{-\sigma}A_0) + (\cdots), \tag{1.62}$$

where Z denotes the partition function of the system, T_0 the temperature at global thermal equilibrium, and $(\cdots)$ gives higher-order terms with spatial derivatives. They call Eq. (1.62) the hydrostatic partition function. We obtain the energy-momentum tensor (the charge current) by taking the variation of $\log Z$ with respect to the metric $g_{\mu\nu}$ (the background gauge field $\mathcal{A}_\mu$):

$$T_{\mu\nu} = -2T_0\frac{\delta}{\delta g^{\mu\nu}}\log Z, \tag{1.63}$$

$$J^\mu = T_0\frac{\delta}{\delta \mathcal{A}_\mu}\log Z. \tag{1.64}$$

Then, matching the functional differentiation (1.63) and (1.64) with the known form of the constitutive relations, they read off the thermodynamic meanings of the parameters such as $T_0 e^{-\sigma}$ and $e^{-\sigma}A_0$. As a result, they reproduce the constitutive relation, which is consistent with the thermodynamic structure. Furthermore, in the presence of the quantum anomaly, this method enables us to derive the anomaly-induced transport. In fact, the first-order derivative correction to $\log Z$, which is consistent with the quantum anomaly, leads to the anomaly-induced transport [41, 85, 86, 87, 88]. Nevertheless, we note that what is assumed in this treatment is rather obscure, similar to the phenomenological approach discussed in the first part of this section. As a consequence of the phenomenological aspect of this derivation, we do not have the Green-Kubo formula for the transport coefficients. In addition, the physical meanings of the background data such as σ, and a_i are not determined without matching the known hydrodynamic equations. Therefore, strictly speaking, this does not provide the derivation of hydrodynamics from underlying microscopic theories. To unravel the reason why this method works is one of the main topics of this thesis, particularly in Chapter 2.

1.2 Anomaly-induced transport phenomena

If the system possesses a certain symmetry, the Noether's theorem tells us an existence of a corresponding conserved quantity [89]. To be exact, this is, however, true only in the classical sense. It is known that some conservation laws fail as a consequence of the quantum correction even if there exist corresponding global symmetries. This phenomenon is known as the quantum anomaly [90, 91, 92]. The vital point concerning the anomaly is that it is associated with the topological properties of gauge theories, and, therefore, exact. When we construct a low-energy effective field theory, we have to properly take into account the quantum anomaly: A well-known example is the Wess-Zumino-Witten term in chiral perturbation theory [93, 94]. It is recently pointed out that the quantum anomaly also affects the macroscopic transport properties in the hydrodynamic regime. In this section, after a short introduction of the quantum anomaly in Sec. 1.2.1, we explain the anomaly-induced transport phenomena such as the chiral magnetic effect [44, 95, 96, 97], the chiral separation effect [98, 99], and the chiral vortical effect [43, 100, 101, 59] in Sec. 1.2.2. In Sec. 1.2.3, we also present some candidate systems where the anomaly-induced transport would take place.

1.2.1 Quantum anomaly

Breakdown of axial current conservation law

Let us consider the massless Dirac fermion in $d = 4$ spacetime dimension, whose Lagrangian reads

$$\mathcal{L} = -\bar{\psi}\gamma^\mu D_\mu \psi, \quad \text{with} \quad D_\mu = \partial_\mu - iA_\mu, \tag{1.65}$$

where γ^μ denote gamma matrices, and $\bar{\psi} \equiv i\psi^\dagger\gamma^0$.

This Lagrangian remains unchanged under the global $U(1)_V$ transformation

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\alpha}, \tag{1.66}$$

where α is the arbitrary constant. The $U(1)_V$ symmetry of the Lagrangian (1.65) brings about the conservation law of the vector current

$$\partial_\mu J^\mu = 0, \quad \text{with} \quad J^\mu = i\bar{\psi}\gamma^\mu\psi. \tag{1.67}$$

Although this is the classical argument, this remains true even if we consider quantum correnctions.

In addition to the $U(1)_V$ symmetry, the Lagrangian (1.65) has another symmetry under the global $U(1)_A$ transformation

$$\psi \to e^{i\gamma_5\alpha}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\gamma_5\alpha}, \tag{1.68}$$

where we introduced the chiral matrix $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$ defined in the case of $d = 4$. Then, expected conservation law associated with the $U(1)_A$ symmetry is

$$\partial_\mu J^\mu_5 = 0 \quad \text{with} \quad J^\mu_5 = i\bar{\psi}\gamma^\mu\gamma_5\psi. \tag{1.69}$$

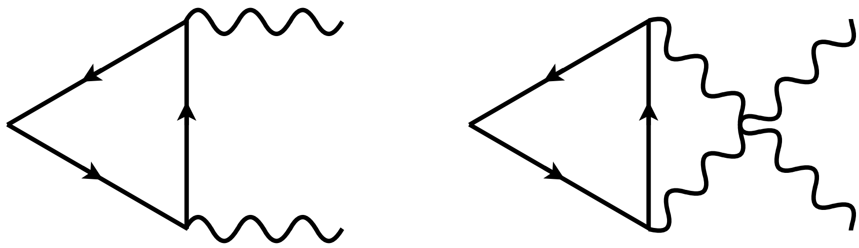


Figure 1.1: Feynman diagrams which contributes to a divergence of the axial current.

However, we find that this is not true if we consider the quantum correction shown in Fig. 1.1. The proper consideration shows that, instead of Eq. (1.67), we obtain

$$\partial_\mu \hat{J}_5^\mu = C_{\text{ano}} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{1.70}$$

where $F_{\mu\nu}$ is the field strength tensor of the electromagnetic fields. Therefore, we have the breakdown of the conservation law of the axial current. This is the quantum anomaly [90, 91, 92].

1.2.2 Anomaly-induced transport

Chiral Magnetic Effect and Chiral Separation Effect

When systems composed of charged particles are put under the electric field $\boldsymbol{E}$, we have the electric current along the electric field

$$\boldsymbol{J}_V = \sigma \boldsymbol{E}, \tag{1.71}$$

where σ denotes the electric conductivity. This is nothing but Ohm's law. An actual value of the electric conductivity is obtained by the use of the Green-Kubo formula, or the Boltzmann equation, and depends on the coupling strength. For example, based on a perturbative calculation, the electric conductivity at high temperature in $U(1)$ gauge theories is evaluated as [102, 103]

$$\sigma = \frac{2.50}{\alpha \log(1.46/\alpha)} T, \tag{1.72}$$

where α is the fine structure constant.

Then, let us consider systems under the magnetic field $\boldsymbol{B}$. We usually do not have the electric current along magnetic fields such as

$$\boldsymbol{J}_V = \sigma_B \boldsymbol{B}, \tag{1.73}$$

where we defined the magnetic conductivity σ_B. This is easily understood with the help of parity symmetry. Under the parity transformation, the electric current changes the sign: $\boldsymbol{J}_V \to -\boldsymbol{J}_V$. On the other hand, the magnetic field is pseudo-vector, and it is unchanged, $\boldsymbol{B} \to \boldsymbol{B}$. Therefore, for Eq. (1.73) to hold with a non-zero current, the magnetic conductivity has to behave like pseudo-scalar, $\sigma_B \to -\sigma_B$ under the parity transformation. However, if the system considered has parity symmetry, we do not have pseudo-scalar quantities, and the magnetic conductivity vanishes: $\sigma_B = 0$.

The situation is drastically changed if the system does not have parity invariance. For instance, let us consider the system composed of the massless fermion with chirality imbalance between left- and right-handed fermions. Then, we can introduce the chiral chemical potential μ_5 which represents the chemical potential for the axial charge $\mu_5 \equiv (\mu_R - \mu_L)/2$, where μ_R (μ_L) denotes the chemical potential for right- (left-) handed fermions. Since the chiral chemical potential is a pseudo-scalar quantity, we can have a current along the magnetic field

$$\boldsymbol{J}_V = \frac{\mu_5}{2\pi^2}\boldsymbol{B}. \tag{1.74}$$

This is the chiral magnetic effect (see Fig. 1.2 (a)) [44, 95, 96, 97]. Comparison between Eq. (1.73) and Eq. (1.74) allows us to find that the chiral magnetic conductivity is given by

$$\sigma_B = \frac{\mu_5}{2\pi^2}. \tag{1.75}$$

Here an important point is that, unlike Eq. (1.72), the chiral magnetic conductivity σ_B does not depend on the coupling constant α, and it is completely determined by the thermodynamic quantity μ_5. This is the important difference between the usual conducting current and the chiral magnetic current. This property arises from the fact that the chiral magnetic effect belongs to a family of the anomaly-induced transport.

We also have the corresponding current for the axial current $\boldsymbol{J}_A$, the so-called chiral separation effect [98, 99], although we do not have the conducting current in this sector. Let us consider a similar situation with the vector charge chemical potential μ. In this case, the chiral separation effect is given by

$$\boldsymbol{J}_A = \frac{\mu}{2\pi^2}\boldsymbol{B}. \tag{1.76}$$

The chiral separation conductivity is also determined only by the chemical potential, and not dependent on the coupling constant. This is because the chiral separation effect is also one example of the anomaly-induced transport.

Chiral Vortical Effect

There is another example of the anomaly-induced transport, the so-called chiral vortical effect. After a long time had passed since the first encounter in Refs. [43], the chiral vortical effect in hydrodynamics has been first recognized based on the fluid/gravity correspondence [100, 101], and has later been confirmed based on the phenomenological derivation of hydrodynamics [59], and also other ways such as the linear response theory [104, 105, 106], and the hydrostatic partition function method [41, 85, 86, 87, 88]. The chiral vortical effect is a phenomenon analogous to the chiral magnetic effect obtained by replacing the magnetic field $\boldsymbol{B}$ by the fluid

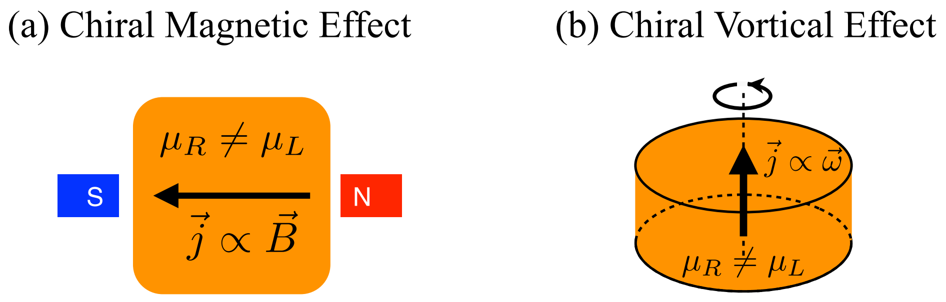


Figure 1.2: Some example of the anomaly-induced transport phenomena. (a) Chiral magnetic effect (b) Chiral vortical effect.

vorticity $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$ with a fluid velocity $\boldsymbol{v}$. The chiral vortical effects for the massless QED are given by

$$\boldsymbol{J}_V = \frac{\mu\mu_5}{2\pi^2}\boldsymbol{\omega}, \quad \boldsymbol{J}_A = \left(\frac{\mu^2 + \mu_5^2}{4\pi^2} + \frac{T^2}{12}\right)\boldsymbol{\omega}, \tag{1.77}$$

where T denotes temperature of the system (see Fig. 1.2 (b)). There is, of course, a similarity that the chiral vortical conductivity does not depend on the coupling constant, but also a difference that it depends on temperature. This suggests that we have different vortical currents, from different physical origins. In fact, some parts come from the chiral anomaly while others do not.

1.2.3 Possible realization in physical systems

Quark-Gluon Plasma in heavy-ion collisions

The quark-gluon plasma (QGP), which is composed of deconfined quarks and gluons, is considered to be a candidate where anomaly-induced transport phenomena take place. The QGP is experimentally created by colliding two heavy-ion nuclei at ultra-relativistic energies. Such experiments have been and are being performed at Relativistic Heavy Ion Collider (RHIC) in BNL, and Large Hadron Collider (LHC) in CERN since 2000.

Since two charged nuclei collide at very high energy, extremely strong magnetic fields, as large as 10^{14} T, are created in off-central collisions (see Fig. 1.3 (a)) [107, 108]. Therefore, the vector (axial) current may flow due to the chiral magnetic (separation) effect along the magnetic fields. At present, the evidence in the experiments has been elusive, and numerous works to assess the contributions from anomaly-induced transport are now underway [109, 110, 111, 112, 113, 114].

Weyl semimetal in condensed matter

It is recently pointed out that the anomaly-induced transport also takes place in a special kind of material, the so-called Weyl semimetal which has separated Weyl nodes [115, 116, 117,

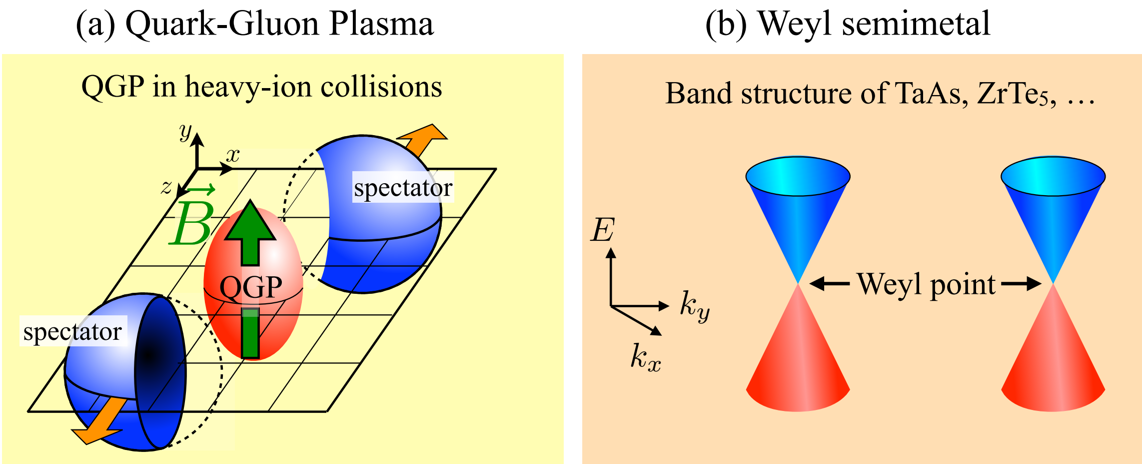


Figure 1.3: Possible physical systems where the anomaly-induced transport take place.

118]. In the vicinity of the Weyl nodes, the dispersion relation of the quasi-patricles becomes approximately linear (see Fig. 1.3 (b)). Therefore, the effective Hamiltonian for these quasi-particles read $H = \pm v_F \boldsymbol{\sigma}\cdot\boldsymbol{k}$, where v_F denotes the Fermi velocity of quasi-particles, σ^i the Pauli matrices, and $\boldsymbol{k}$ the momentum. This is nothing but the Hamiltonian for the Weyl fermions.

The experimental realization of the Weyl semimetal is now under intensive studies. Some experimental signature of the Weyl fermions and the chiral magnetic effect in materials such as TaAs, $ZrTe_5$ have recently been reported [119, 120, 121].

1.3 Outline

This thesis is organized as follows. Chapter 2, we review local thermodynamics and derive the path-integral formulation of the Massieu-Planck functional on a hypersurface. As a result, we show that a thermodynamic potential, or the Masseiu-Planck funcional, is written in terms of quantum field theories in the emergent curved spacetime, whose structure is determined by thermodynamic parameters. This analysis gives a solid basis to derive the nondissipative part of the constitutive relations. In Chapter 3, we consider the time evolution of hydrodynamic variables, and derive the zeroth-order and first-order constitutive relations for normal relativistic fluids[5]. We show that the obtained constitutive relations results in the one for a perfect fluid and for the Navier-Stokes fluid, respectively. In Chapter 4, we consider the parity-violating system under the background electromagnetic fields which contains the quantum anomaly, and derive the anomaly-induced transport phenomena from quantum field theories. Based on the formulation presented in Chapter 2, we calculate the Masseiu-Planck functional in the presence of the background electromagnetic fields and background "gravitational" fields with the help

[5] By "normal" fluids, we mean fluids which do not have neither spontaneously broken symmetries nor the parity violation.

of the perturbative method. Chapter 5 is devoted to conclusions and outlook. The overall structure of this thesis is presented in Fig. 1.4.

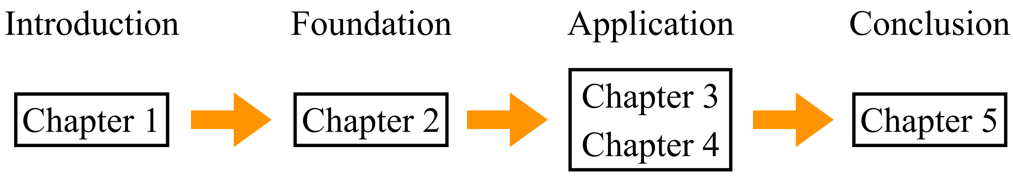


Figure 1.4: Overall structure of this thesis.

1.4 Notation

In this thesis, we use the natural unit,

$$c = \hbar = k_B = 1. \tag{1.78}$$

We use Greek letters $\mu, \nu, \cdots$ for the d-dimensional curved spacetime indices, which run over $0, 1, 2, \cdots, d-1$ with x^0 the time coordinate. Latin letters starting from $i, j, \cdots$ run over the $d-1$-dimensional Euclidean space coordinate labels $1, 2, \cdots, d-1$, while we use ones starting from $a, b, \cdots$ for the d-dimensional (local) Minkowski indices[6].

We adopt the mostly positive convention for the metric,

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \cdots, 1), \tag{1.79}$$

and also adopt the convention that $\varepsilon^{\mu_0\mu_1\cdots\mu_{d-1}}$ is the totally antisymmetric tensor with

$$\varepsilon^{012\cdots d-1} = 1. \tag{1.80}$$

In Chapter 4, we introduce the imaginary-time notation

$$X^\mu \equiv (\tau, \boldsymbol{x}), \tag{1.81}$$

where τ denotes the imaginary time appeared in the thermal field theory, and $\boldsymbol{x}$ the spatial vector. Fourier transformation in the imaginary-time formalism is carried out as

$$\varphi(X) = T \sum_{\omega_n} \int \frac{d^{d-1}x}{(2\pi)^{d-1}} e^{-i\omega_n \tau + i\boldsymbol{p}\cdot\boldsymbol{x}} \varphi(\omega_n, \boldsymbol{p}) \equiv \sum_P e^{iP\cdot X} \varphi(P), \tag{1.82}$$

where T is the global temperature of systems, ω_n the Matsubara frequency, and P^μ is introduced as

$$P^\mu \equiv (i\omega_n, \boldsymbol{p}). \tag{1.83}$$

[6] These (local) Minkowski indices are used to describe spinor fields in the curved spacetime in the subsequent chapters.

Chapter 2

Quantum field theory for locally thermalized system

In this chapter, extending the imaginary-time formalism of the finite temperature quantum field theory, we formulate quantum field theories for locally thermalized systems. In particular, we derive the path-integral formula for the Massieu-Planck functional $\Psi = \log Z$, where Z is a hydrodynamic partition function. In the same way as the finite temperature field theory for globally thermalized systems [122, 123, 124], the Massieu-Planck functional is written in terms of the path integral of a Euclidean action. It is, however, not the Euclidean action in flat spacetime but one in the emergent curved spacetime background, whose metric depends on thermodynamic parameters such as the local temperature and fluid four-velocity [1]. This arises from inhomogeneity of the local Gibbs distribution, which describes locally thermalized systems. Performing the explicit path-integral analysis for the Masseiu-Planck functional, we show that this emergent spacetime has useful symmetries such as Kaluza-Klein gauge symmetry, spatial diffeomorphism symmetry, and gauge symmetry for external fields. These symmetry properties are in accordance with those of recent studies, in which the Massieu-Planck functional is deduced on the basis of symmetry and scaling properties [41, 42]. Symmetry properties for the Massieu-Planck functional give a solid basis to derive relativistic hydrodynamic equations in the subsequent chapters.

This chapter is organized as follows: In Sec. 2.1, we briefly review the finite temperature field theory, especially the imaginary-time formalism, which enables us to calculate thermodynamic properties based on the quantum field theory. In Sec. 2.2, we focus on the local thermodynamics on a hypersurface, and introduce the local Gibbs distribution. We also show that the Masseiu-Planck functional is regarded as the generating functional for nondissipative hydrodynamics. In Sec. 2.3, we give the path-integral formulation of the Massieu-Planck functional on a hypersurface for representative examples of quantum fields. In Sec. 2.4, we show symmetry properties of the emergent thermal spacetime which the Massieu-Planck functional respects. Sec. 2.5 is devoted to a short summary of this chapter.

The formulation and whole works presented from Sec. 2.2 to Sec. 2.5 are based on our original work in collaboration with Yoshimasa Hidaka (RIKEN). The part of them are also in collaboration with Yoshimasa Hidaka (RIKEN), Tomoya Hayata (RIKEN), and Toshifumi Noumi (Hong Kong University of Science and Technology) [1].

2.1 Review on finite temperature field theory

Basics of quantum statistical mechanics in thermal equilibrium

To study the system under thermal equilibrium, we start with an appropriate statistical ensemble [122, 123, 124]. Here we introduce the so-called Gibbs ensemble [45, 46], or grand canonical ensemble, which is described by the density operator

$$\hat{\rho}_G = \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{Z} \quad \text{with} \quad Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}, \tag{2.1}$$

where $\hat{H}$ and $\hat{N}$ denote Hamiltonian and a conserved total charge operator, respectively, while β and μ denote inverse temperature and chemical potential, respectively. The partition function Z gives the normalization factor of the probability distribution. This Gibbs ensemble is understood as the statistical ensemble with the maximum information entropy $S = -\mathrm{Tr}\,\hat{\rho}\log\hat{\rho}$ under the constraints that the average values of total energy and conserved charge are fixed: $\langle\hat{H}\rangle = E = \text{const.}$ and $\langle\hat{N}\rangle = N = \text{const.}$ The expectation value of operators is given by the thermal average over the Gibbs distribution $\hat{\rho}_G(\beta,\mu)$

$$\langle\hat{\mathcal{O}}(x)\rangle_G = \mathrm{Tr}\,\hat{\rho}_G\hat{\mathcal{O}}(x). \tag{2.2}$$

Equilibrium statistical mechanics tells us that the thermodynamic potential constructed from the partition function Z contains enough information on thermal properties of systems such as the specific heat, charge susceptibility, and the equation of state. In fact, once we evaluate the following thermodynamic function[1]

$$\Psi(\beta,\nu,V) = \log Z(\beta,\nu,V), \tag{2.3}$$

where V denotes the volume of the system and $\nu \equiv \beta\mu$, we can obtain the expectation value of the total energy E, and conserved charge N by

$$E \equiv \langle\hat{H}\rangle_G = -\frac{\partial\Psi}{\partial\beta}, \quad N \equiv \langle\hat{N}\rangle_G = \frac{\partial\Psi}{\partial\nu}, \quad \beta p \equiv \frac{\partial\Psi}{\partial V}, \tag{2.4}$$

where p denotes the pressure. Then, the first law of thermodynamics is given by

$$d\Psi = -Ed\beta + Nd\nu + \beta pdV. \tag{2.5}$$

[1] According to Ref. [125], this thermodynamic function is called as the Kramers function. However, when we consider local thermal equilibrium in the subsequent chapters, we call the generalization of this thermodynamic function as the Masseiu-Planck functional.

Performing the Legendre transformation, we can also describe the thermodynamic relation in an alternative way. By defining the entropy,

$$S \equiv -\operatorname{Tr} \hat{\rho}_G \log \hat{\rho}_G = \beta E - \nu N + \Psi, \tag{2.6}$$

we easily see the first law of thermodynamics,

$$dS = \beta dE - \nu dN + \beta p dV. \tag{2.7}$$

Furthermore, dividing all the extensive quantities by the volume V such as $\psi \equiv \Psi/V = pV$, $e \equiv E/V$, $n \equiv N/V$, and $s \equiv S/V = \beta e - \nu n + \psi$, we obtain the following expressions of the first law of thermodynamics

$$d\psi = -e d\beta + n d\nu, \quad ds = \beta de - \nu dn. \tag{2.8}$$

Imaginary-time formalism for finite temperature field theory

In order to study thermodynamic properties of systems based on the quantum field theory, we have well-established formalisms such as imaginary-time formalism [126, 127], real-time formalism [128, 129, 123], and thermo-field dynamics [130, 131]. These formalisms look very different from each other, but give equivalent results in thermal equilibrium. Here we focus on the imaginary-time formalism, which plays an important role in this thesis.

In the imaginary-time formalism, or the Matsubara formalism, we start from the Lagrangian in the Minkowski space $\mathcal{L}(\varphi, \partial_\mu \varphi)$, and construct the Hamiltonian $\hat{H}(\hat{\varphi}, \hat{\pi})$, and the conserved charge $\hat{N}(\hat{\varphi}, \hat{\pi})$, where φ and π denotes fields and its canonical momentum. Then, we express the partition function as

$$Z = \operatorname{Tr} e^{-\beta(\hat{H} - \mu \hat{N})} = \int d\varphi \langle \pm\varphi | e^{-\beta(\hat{H} - \mu \hat{N})} | \varphi \rangle, \tag{2.9}$$

where $\pm$ corresponds to the case of bosonic or fermionic fields: $+$ for bosonic fields, and $-$ for the fermionic fields. Recalling that the path-integral representation of the transition amplitude in real-time from a state $|\varphi_a\rangle$ at time $t = t_i$ to a state $|\varphi_b\rangle$ at time $t = t_f$ is given by

$$\begin{aligned} K_{ba}(t_f, t_i) &= \langle \varphi_b | e^{-i\hat{H}(t_f - t_i)} | \varphi_a \rangle \\ &= \int_{\varphi(t_1) = \varphi_a}^{\varphi(t_f) = \varphi_b} \mathcal{D}\varphi \, \exp\left[i \int_{t_i}^{t_f} dt \int d^{d-1}x \, \mathcal{L}(\varphi, \partial_\mu \varphi) \right], \end{aligned} \tag{2.10}$$

Comparing Eq. (2.9) with Eq. (2.10), we immediately find that the partition function Z is obtained by taking a summation over state φ after the following replacement in Eq. (2.10): $\hat{H} \to \hat{H} - \mu \hat{N}$, $i(t_f - t_i) \to \beta$, and $\varphi_a = \pm\varphi_b \to \varphi$. As a result, we obtain the path-integral formula for the partition function,

$$Z = \int_{\varphi(\beta) = \pm\varphi(0)} \mathcal{D}\varphi \, e^{+S_E[\varphi]}, \quad \text{with} \quad S_E[\varphi] = \int_0^\beta d\tau \int d^{d-1}x \, \mathcal{L}_E(\varphi, \partial_\mu \varphi), \tag{2.11}$$

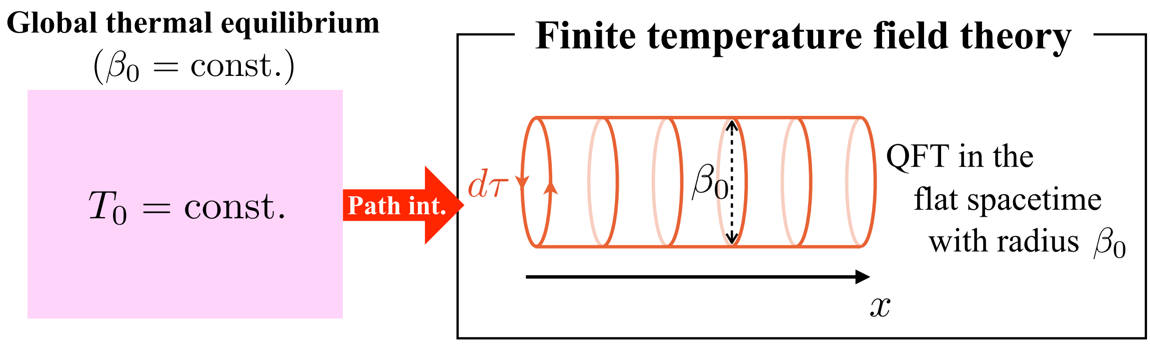


Figure 2.1: Schematic picture of the imaginary-time formalism.

where φ denotes a quantum field dependent on the imaginary-time, and $S_E[\varphi]$ is the Euclidean action which is obtained by replacing $it \to \tau$ in the original Lagrangian, and restricting the integration range for imaginary time as $\tau_i = 0$ to $\tau_f = \beta$. The boundary condition for the fields in the temporal direction depends on the spin of the fields: periodic, or $\varphi(\tau = 0) = \varphi(\tau = \beta)$ for bosonic fields, and anti-periodic, or $\varphi(\tau = 0) = -\varphi(\tau = \beta)$ for fermionic fields. Therefore, the partition function is obtained by the use of the quantum field theory in the Euclidean flat spacetime (see Fig. 2.1).

2.2 Local thermodynamics on a hypersurface

In this section we discuss the local thermodynamics on a spacelike hypersurface in order to formulate quantum field theories for locally thermalized systems in a covariant way. In Sec. 2.2.1, we summarize geometric aspects of the spatial hypersurface used in this thesis. In Sec. 2.2.2, we derive the conservation law for the system under the external field based on symmetry arguments. In Sec. 2.2.3, we introduce several concepts such as the local Gibbs distribution, and the Masseiu-Planck functional, which play a central role in our derivation of hydrodynamics.

2.2.1 Geometric preliminary

Decomposition of spacetime with hypersurface

As a technical preparation, we first summarize the geometric aspects of spacelike hypersurface in this subsection. Let us consider spatial slicings on a general curved spacetime with a metric $g_{\mu\nu}$ and parametrize the spacelike hypersurface by $\bar{t}$. We also introduce the spatial coordinates $\bar{\boldsymbol{x}}$ on the hypersurface. In other words, we define a spacelike hypersurface $\Sigma_{\bar{t}}$ by the $\bar{t}(x) =$ const. surface, and introduce spatial coordinates $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(x)$, where x is a general coordinate (see Fig. 2.2). To discuss dynamics on such a spacelike hypersurface, it is convenient to introduce a timelike unit vector n_μ as

$$n_\mu(x) = -N(x)\partial_\mu \bar{t}(x) \quad \text{with} \quad N(x) \equiv \left(-\partial^\mu \bar{t}(x)\partial_\mu \bar{t}(x)\right)^{-1/2}. \tag{2.12}$$

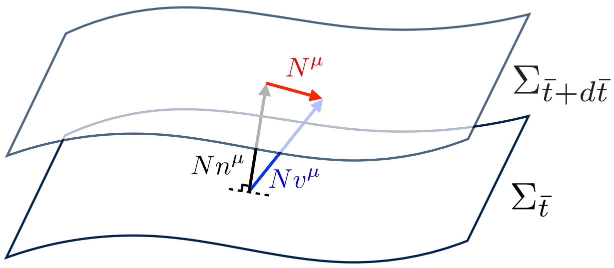


Figure 2.2: Illustration of the Arnowitt-Deser-Misner (ADM) decomposition of the spacetime. $\Sigma_{\bar{t}}$ denotes a spacelike hypersurface parametrized by $\bar{t}(x) = \text{const.}$ n^μ is a vector normal to the hypersurface. Introducing the lapse function $N(x)$ and the shift vector $N^\mu(x)$, we decompose the time vector as $t^\mu \equiv \partial_{\bar{t}} x^\mu = N v^\mu = N n^\mu + N^\mu$. The figure is taken from [1].

Here we normalize n_μ as $n_\mu n^\mu = -1$ and n^μ is future oriented. $N > 0$ is the lapse function. We use the mostly plus convention of the metric, e.g., the Minkowski metric is $\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, \cdots, 1)$. The induced metric $\gamma_{\mu\nu}$ on the spacelike hypersurface is then

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \,. \tag{2.13}$$

We also introduce the time vector $t^\mu \equiv \partial_{\bar{t}} x^\mu(\bar{t}, \bar{\boldsymbol{x}})$, and the shift vector N^μ by the decomposition

$$t^\mu \equiv \partial_{\bar{t}} x^\mu(\bar{t}, \bar{\boldsymbol{x}}) = N n^\mu + N^\mu \quad \text{with} \quad n_\mu N^\mu = 0 \,. \tag{2.14}$$

In the coordinate system $(\bar{t}, \bar{\boldsymbol{x}})$, n_μ, $\gamma_{\mu\nu}$, and N^μ are given explicitly by

$$n_{\bar{\mu}} = (-N, \boldsymbol{0}) \,, \quad \gamma_{\bar{0}\bar{i}} = \gamma_{\bar{i}\bar{0}} = g_{\bar{0}\bar{i}} = g_{\bar{i}\bar{0}} \,, \quad \gamma_{\bar{i}\bar{j}} = g_{\bar{i}\bar{j}} \,, \quad N^{\bar{\mu}} = \begin{pmatrix} 0 \\ N^2 g^{\bar{0}\bar{i}} \end{pmatrix} . \tag{2.15}$$

The metric $g_{\bar{\mu}\bar{\nu}}$ takes the form of the Arnowitt-Deser-Misner (ADM) metric,

$$g_{\bar{\mu}\bar{\nu}} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^{\bar{\mu}}} \frac{\partial x^\nu}{\partial \bar{x}^{\bar{\nu}}} = \begin{pmatrix} -N^2 + N_{\bar{i}} N^{\bar{i}} & N_{\bar{j}} \\ N_{\bar{i}} & \gamma_{\bar{i}\bar{j}} \end{pmatrix} , \qquad g^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} -N^{-2} & N^{-2} N^{\bar{j}} \\ N^{-2} N^{\bar{i}} & \gamma^{\bar{i}\bar{j}} - N^{-2} N^{\bar{i}} N^{\bar{j}} \end{pmatrix} . \tag{2.16}$$

Here $N_{\bar{i}} = \gamma_{\bar{i}\bar{j}} N^{\bar{j}}$. $\gamma^{\bar{i}\bar{j}}$ is the inverse of $\gamma_{\bar{i}\bar{j}}$ and satisfies $\gamma_{\bar{i}\bar{j}} \gamma^{\bar{j}\bar{k}} = \delta_{\bar{i}}^{\bar{k}}$. The d-dimensional volume element is given by

$$\int d^d x \sqrt{-g} = \int d^d x N \sqrt{\gamma} \quad \text{with} \quad \gamma = \det \gamma_{\bar{i}\bar{j}} \,, \tag{2.17}$$

whereas the volume element on the spacelike hypersurface $\Sigma_{\bar{t}}$ is

$$\int d\Sigma_{\bar{t}} = \int d^d x \sqrt{-g}\, \delta\big(\bar{t} - \bar{t}(x)\big) N^{-1}(x) = \int d^{d-1} \bar{x} \sqrt{\gamma} \,. \tag{2.18}$$

It is also convenient to introduce a vector v^μ proportional to Eq. (2.14) as

$$v^\mu = N^{-1} t^\mu \quad \text{with} \quad v^\mu n_\mu = -1 \,. \tag{2.19}$$

Using n_μ and v^μ, we define a spatial projection operator P^μ_ν as

$$P^\mu_\nu \equiv \delta^\mu_\nu + v^\mu n_\nu \quad \text{with} \quad P^\mu_\nu v^\nu = 0\,, \quad P^\mu_\nu n_\mu = 0\,, \quad P^\mu_\rho P^\rho_\nu = P^\mu_\nu\,. \tag{2.20}$$

Its concrete form in the coordinate system $(\bar{t}, \bar{\boldsymbol{x}})$ is given by $P^{\bar{\mu}}_{\bar{\nu}} = \mathrm{diag}(0, 1, 1, \ldots, 1)$. We will use this projection operator in Sec. 3.2.1. We note that such an operator often appears in the context of Newton-Cartan geometry (see, e.g., Refs. [132, 133][2]).

2.2.2 Matter field

Energy-momentum conservation law under external field

Following the geometric preliminary, we next consider the matter sector and set out a general relation between symmetries of system and conservation laws[3]. We consider matter actions in a general curved spacetime background $g_{\mu\nu}$ and an external gauge field A_μ, which is given by

$$S[\varphi; g_{\mu\nu}, A_\mu] = \int d^d x \sqrt{-g} \mathcal{L}(\varphi_i(x), \partial_\mu \varphi_i(x); g_{\mu\nu}(x), A_\mu(x)), \tag{2.21}$$

where φ_i denotes a set of matter fields under consideration, and spacetime integral runs within all region in which matter fields take place[4]. Here we consider general situations with charged matter fields, but if matter fields are not charged, we do not have the external gauge field.

Since the action remains invariant under a general coordinate transformation, we have corresponding conserved charge currents associated with diffeomorphism invariance. Let us consider the following infinitesimal coordinate transformation,

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x), \tag{2.22}$$

where $\xi^\mu(x)$ denotes an arbitrary infinitesimal vector. We assume that $\xi^\mu(x)$ vanishes on the boundary of the region of spacetime integration for the action. Under the infinitesimal cooridinate transformation (2.22), variations of the metric $g_{\mu\nu}$, the external gauge field A_μ, and matter fields φ_i which are nothing but Lie derivatives along ξ^μ, are given by

$$\delta_\xi g_{\mu\nu} \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{2.23}$$

$$\delta_\xi A_\mu \equiv A'_\mu(x) - A_\mu(x) = \xi^\nu \nabla_\nu A_\mu + A_\nu \nabla_\mu \xi^\nu, \tag{2.24}$$

$$\delta_\xi \varphi_i \equiv \varphi'_i(x) - \varphi_i(x), \tag{2.25}$$

[2] Our normalization $n_\mu v^\mu = -1$ has the opposite sign compared to that in Refs. [132, 133]

[3] Here, we do not consider quantum anomalies, which brings about the breakdown of the conservation law in spite of the existence of classical symmetries. We will discuss the effects of quantum anomalies on hydrodynamics in Chap. 4.

[4] If the action consists of spinor fields, it is not written in terms of the metric. Therefore, we have to take a slightly different way. Due to the extensive preparation required for that case, it will be discussed when we consider the Dirac field in Sec. 2.3.3

where the explicit form of $\delta_\xi \varphi_i$ depends on the spin of fields such as $\delta_\xi \phi = \xi^\mu \partial_\mu \phi$ for the scalar field, and $\delta_\xi B_\mu = \xi^\nu \nabla_\nu B_\mu + B_\nu \nabla_\mu \xi^\mu$ for the vector field. We, however, get rid of a change invoked by the variation of fields φ_i, with the help of the equation of motion for φ_i: $\delta S/\delta\varphi_i = 0$. Therefore, we obtain an expression for the variation of the action,

$$\begin{aligned}\delta S &= \int d^d x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \delta_\xi g_{\mu\nu} + J^\mu \delta_\xi A_\mu \right] \\ &= \int d^d x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) + J^\mu (\xi^\nu \nabla_\nu A_\mu + A_\nu \nabla_\mu \xi^\nu) \right] \\ &= -\int d^d x \sqrt{-g} \left[(\nabla_\mu T^\mu_{\ \nu} - F_{\nu\lambda} J^\lambda) \xi^\nu \right] + \int d^d x \sqrt{-g} \nabla_\mu [(T^\mu_{\ \nu} + J^\mu A_\nu)\xi^\nu] - \int d^d x \sqrt{-g} \xi^\nu A_\nu \nabla_\mu J^\mu,\end{aligned} \tag{2.26}$$

where we defined the energy-momentum tensor $T^{\mu\nu}$ and charge current J^μ by taking variations of the action with respect to the metric and external gauge field

$$T^{\mu\nu}(x) \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}, \quad J^\mu(x) \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta A_\mu(x)}. \tag{2.27}$$

Here we also introduced a field strength tensor of the external gauge field $F_{\mu\nu}$ as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.28}$$

The second term in the last line of Eq. (2.26) vanishes because it gives an integration on the boundary of the region, where ξ^μ does not take values. Furthermore, the third term also vanishes due to the conservation law for the charge current as will be explained below. Since the action is invariant ($\delta S = 0$) under the above transformation with an arbitrary $\xi^\mu(x)$, Eq. (2.26) results in the energy-momentum conservation law under the external field,

$$\nabla_\mu T^\mu_{\ \nu} = F_{\nu\lambda} J^\lambda. \tag{2.29}$$

We note that this energy-momentum tensor is symmetric under $\mu \leftrightarrow \nu$ by definition[5]. This energy-momentum conservation law is an essential piece for hydrodynamics.

Charge conservation law

As is already mentioned, if systems include charged matter fields, the charge current J^μ is also conserved. This stems from an internal symmetry of the action. Since this internal symmetry of the action is gauged, and is indeed written in terms of the covariant derivative, the action possesses gauge invariance under the $U(1)$ gauge transformation

$$\delta_\alpha A_\mu = \partial_\mu \alpha, \tag{2.30}$$

$$\delta_\alpha \varphi_i = i c_i \alpha \varphi_i, \tag{2.31}$$

[5]When we consider spinor fields, we use a vielbein $e_\mu^{\ a}$ instead of the metric $g_{\mu\nu}$. We note that symmetry under $\mu \leftrightarrow \nu$ is not obvious in such a case. This is discussed in Sec. 2.3.3. Also see the discussion on the Noether current in Appendix B.

where $\alpha(x)$ is an infinitesimal arbitrary function, which is assumed to be zero on the boundary, and $c_i = \pm 1,\ 0$ denotes the charge of matter fields φ_i. The action is invariant under this $U(1)$ gauge transformation: $\delta S = 0$.

We would like to express the variation of the action. Here we do not have to consider the variation of the matter fields, because it does not contribute to the variation of the action by the use of the equation of motion: $\delta S/\delta\varphi_i = 0$. Therefore, the variation of action is given by

$$\begin{aligned} \delta S &= \int d^d x \sqrt{-g} J^\mu \delta_\alpha A_\mu = \int d^d x \sqrt{-g} J^\mu \partial_\mu \alpha \\ &= -\int d^d x \sqrt{-g}\, [(\nabla_\mu J^\mu)\alpha] + \int d^d x \sqrt{-g} \nabla_\mu (J^\mu \alpha). \end{aligned} \tag{2.32}$$

The second term in the second line of Eq. (2.118) is the boundary term, and again vanishes. Since this holds for arbitrary $\alpha(x)$, we obtain the resulting conservation law

$$\nabla_\mu J^\mu = 0, \tag{2.33}$$

where J^μ is defined by the variation of the action with respect to the external gauge field in Eq. (2.27).

2.2.3 Local Gibbs distribution and Masseiu-Planck functional

Local Gibbs distribution

We next introduce a density operator representing a local thermal equilibrium state, and review the thermodynamics on the hypersurface [56, 134, 135, 57]. We start with global thermal equilibrium on the Minkowski space, in which the density operator for an arbitrary inertial frame of reference is given as the Gibbs distribution,

$$\hat{\rho}_{\mathrm{eq}}(\beta^\mu, \nu) = e^{\beta^\mu \hat{P}_\mu + \nu \hat{N} - \Psi(\beta^\mu, \nu)}, \tag{2.34}$$

where parameters are $\beta^\mu = \beta u^\mu$ with the inverse temperature β, the fluid four-velocity of the system u^μ normalized by $u^\mu u_\mu = -1$, and $\nu = \beta\mu$ with the chemical potential μ. $\hat{P}_\mu$ and $\hat{N}$ denote energy-momentum and number operators, respectively. The Massieu-Planck function $\Psi(\beta^\mu, \nu) \equiv \log \mathrm{Tr} \exp[\beta^\mu \hat{P}_\mu + \nu \hat{N}]$ determines the normalization of the density operator $\hat{\rho}_{\mathrm{eq}}$. At the rest frame of medium, $u^\mu = (1, \mathbf{0})$, and thus $\hat{\rho}_{\mathrm{eq}}(\beta^\mu, \nu) = \exp\bigl[-\beta(\hat{H} - \mu\hat{N}) - \Psi(\beta, \nu)\bigr]$ are satisfied. The Gibbs distribution is regarded as a probability distribution obtained by maximizing the entropy $S \equiv -\mathrm{Tr}\,\hat{\rho}\log\hat{\rho}$ under the constraint that the average of total energy and charge are fixed: $\mathrm{Tr}\,\hat{\rho}\hat{H} = \text{const.}$, and $\mathrm{Tr}\,\hat{\rho}\hat{N} = \text{const.}$ In fact, we can interpret that β and ν as Lagrange multipliers, which fix the average energy and charge.

We then generalize the global Gibbs distribution (2.34) to a local form in a coordinate-invariant way. For this purpose, let us consider local thermodynamics on the spacelike hypersurface, $\Sigma_{\bar{t}}$, introduced in the previous subsection. For generality, we leave the metric $g_{\mu\nu}$ of the spacetime as a general curved one. We can construct the density operator which reproduces local thermodynamics on a given hypersurface $\Sigma_{\bar{t}}$, by maximizing the entropy functional $S \equiv -\operatorname{Tr}\hat{\rho}\log\hat{\rho}$ under the constraint that the conserved charge densities are fixed. Since we put the constraint on the average values of the local charge densities, the corresponding Lagrange multipliers are also local one. As a result, we introduce a local Gibbs distribution $\hat{\rho}_{\mathrm{LG}}[\bar{t};\lambda]$ on the hypersurface as

$$\hat{\rho}_{\mathrm{LG}}[\bar{t};\lambda] \equiv \exp\bigl(-\hat{S}[\bar{t};\lambda]\bigr) \quad \text{with} \quad \hat{S}[\bar{t};\lambda] \equiv \hat{K}[\bar{t};\lambda] + \Psi[\bar{t};\lambda]\,, \tag{2.35}$$

where $\hat{K}[\bar{t};\lambda]$ is defined by

$$\hat{K}[\bar{t};\lambda] \equiv -\int d\Sigma_{\bar{t}\mu}\,\lambda^a(x)\hat{\mathcal{J}}^\mu_a(x) = -\int d\Sigma_{\bar{t}\nu}\Bigl(\beta^\mu(x)\hat{T}^\nu{}_\mu(x) + \nu(x)\hat{J}^\nu(x)\Bigr). \tag{2.36}$$

Here we introduced $d\Sigma_{\bar{t}\mu} = -d\Sigma_{\bar{t}} n_\mu$. λ^a and $\hat{\mathcal{J}}^\mu_a$ denote sets of parameters, $\lambda^a(x) \equiv \{\beta^\mu(x), \nu(x)\}$, and of current operators, $\hat{\mathcal{J}}^\mu_a(x) \equiv \{\hat{T}^\mu{}_\nu(x), \hat{J}^\mu(x)\}$, respectively. Just as in the global case (2.34), the Massieu-Planck functional $\Psi[\bar{t};\lambda]$ determines the normalization of the density operator $\hat{\rho}_{\mathrm{LG}}$,

$$\Psi[\bar{t};\lambda] \equiv \log\operatorname{Tr}\exp(-\hat{K}[\bar{t};\lambda]). \tag{2.37}$$

For constant parameters and $n_\mu = (-1,\mathbf{0})$, the local Gibbs distribution reproduces the global one (2.34). We note that the definition here is coordinate invariant by construction.

The charge density operators on the hypersurface, $\hat{c}_(x) = \{\hat{p}_\mu(x), \hat{n}'(x)\}$[6], read $\hat{p}_\mu(x) \equiv -n_\nu(x)\hat{T}^\nu{}_\mu(x)$ and $\hat{n}'(x) \equiv -n_\nu(x)\hat{J}^\nu(x)$. Their expectation values, $\langle\hat{c}_a(x)\rangle^{\mathrm{LG}}_{\bar{t}} \equiv \operatorname{Tr}[\hat{\rho}_{\mathrm{LG}}[\bar{t};\lambda]\hat{c}_a(x)]$, are obtained from the variation of the Masseiu-Planck functional $\Psi[\bar{t};\lambda]$ with respect to the thermodynamic parameters $\lambda^a(x)$ on $\Sigma_{\bar{t}}$,

$$c_a(x) \equiv \langle\hat{c}_a(x)\rangle^{\mathrm{LG}}_{\bar{t}} = \frac{\delta}{\delta\lambda^a(x)}\Psi[\bar{t};\lambda]. \tag{2.38}$$

Therefore, when the Masseiu-Planck functional is known as a functional of the thermodynamic parameters λ^a, it enables us to extract all the thermodynamic properties like the equation of state. This is the reason why the Masseiu-Planck functional belongs to the family of the thermodynamic potentials.

For later purposes, we introduce the Masseiu-Planck current ψ^μ such that

$$\Psi[\bar{t};\lambda] = \int d\Sigma_{\bar{t}\mu}\psi^\mu = \int d\Sigma_{\bar{t}}\psi, \tag{2.39}$$

where $\psi = -n_\mu\psi^\mu$, which satisfies

$$d\psi = c_a d\lambda^a = p_\mu d\beta^\mu + n' d\nu, \tag{2.40}$$

[6] Note that Latin letters on thermodynamic parameters $c_a(x)$ and $\lambda^a(x)$ and current density $\hat{\mathcal{J}}^\mu_a(x)$ do not denote the (local) Lorentz indices to describe spinor fields.

up to the covariant total derivative that does not contribute to $\delta\Psi$. As will be seen in Sec. 3.3, in the leading order of derivative expansion, we can write ψ^μ as $\psi^\mu = \beta^\mu p(\beta, \nu)$ with the pressure p. We note that there is an ambiguity in the definition of ψ^μ because Ψ is invariant under the transformation $\psi^\mu \to \psi^\mu + g^\mu$ with a function g^μ satisfying $n_\mu g^\mu = 0$.

Performing the Legendre transformation, we can also describe the local thermodynamics by the use of the entropy functional. The entropy functional is defined by

$$\begin{aligned} S[\bar{t}; c] &\equiv -\operatorname{Tr} \hat{\rho}_{\mathrm{LG}}[\bar{t}; \lambda] \log \hat{\rho}_{\mathrm{LG}}[\bar{t}; \lambda] \\ &= \langle \hat{S}[\bar{t}; \lambda] \rangle_{\bar{t}}^{\mathrm{LG}} \\ &= -\int d\Sigma_{\bar{t}} \lambda^a c_a + \Psi[\bar{t}; \lambda], \end{aligned} \tag{2.41}$$

where the entropy operator $\hat{S}[\bar{t}; \lambda]$ is defined in Eq. (2.35). The entropy is not a functional of thermodynamic parameters λ^a, but conserved densities c_a, which can be confirmed by conducting the variation of S with fixed $\bar{t}$,

$$\begin{aligned} \delta S &= \int d\Sigma_{\bar{t}} \Big(-\delta\lambda^a c_a - \lambda^a \delta c_a + \frac{\delta \Psi[\bar{t}; \lambda]}{\delta \lambda^a} \delta \lambda^a \Big) \\ &= -\int d\Sigma_{\bar{t}} \lambda^a \delta c_a . \end{aligned} \tag{2.42}$$

Therefore, the thermodynamic parameters are obtained as functional derivatives of entropy with respect to c_a

$$\lambda^a(x) = -\frac{\delta}{\delta c_a(x)} S[\bar{t}; c]. \tag{2.43}$$

This formula again enables us to extract all the thermodynamic properties of the system once the dependence on the conserved charge densities of the entropy functional is acquired. As a result, the entropy functional is counted among the family of the thermodynamic potentials.

Here we introduce the entropy current operator as

$$\hat{s}^\mu \equiv -\lambda^a \hat{\mathcal{J}}^\mu_a + \psi^\mu = -\beta^\nu \hat{T}^\mu_{\ \nu} - \nu \hat{J}^\mu + \psi^\mu, \tag{2.44}$$

the entropy reads

$$S = \int d\Sigma_{\bar{t}\mu} s^\mu = \int d\Sigma_{\bar{t}} s\,, \tag{2.45}$$

where $s^\mu \equiv \langle \hat{s}^\mu \rangle_{\bar{t}}^{\mathrm{LG}}$, and $s = -n_\mu s^\mu = -\lambda^a c_a + \psi = -\beta^\mu p_\mu - \nu n' + \psi$. The entropy density s satisfies the thermodynamic relation, $ds = -\lambda^a dc_a = -\beta^\mu dp_\mu - \nu dn'$, up to the covariant total derivative. These concepts related to the entropy play important roles when we consider the time evolution in Chap. 3.

Masseiu-Planck functional as generating functional under special gauge choice

Here we derive a valuable formula which relates the Masseiu-Planck functional to the average value of the conserved current operators such as $\hat{T}^{\mu\nu}(x)$ and $\hat{J}^{\mu}(x)$ over the local Gibbs distribution. For sake of simplicity, we restrict a discussion to a specific choice of the coordinate system though we have such a relation without choosing the special coordinate system [136].

Let us consider the derivative of $\Psi[\bar{t},\lambda]$ with respect to $\bar{t}$, which reads

$$\begin{aligned}\partial_{\bar{t}}\Psi[\bar{t};\lambda] &= -\langle \partial_{\bar{t}}\hat{K}[\bar{t};\lambda]\rangle^{\mathrm{LG}}_{\bar{t}} \\ &= \left\langle \partial_{\bar{t}} \int d\Sigma_{\bar{t}\mu}\lambda^a \hat{\mathcal{J}}^{\mu}_a \right\rangle^{\mathrm{LG}}_{\bar{t}} \\ &= \left\langle \int d\Sigma_{\bar{t}} N\nabla_\mu\big(\lambda^a \hat{\mathcal{J}}^{\mu}_a\big) \right\rangle^{\mathrm{LG}}_{\bar{t}} \\ &= \int d\Sigma_{\bar{t}} N(\nabla_\mu \Lambda^a)\langle \hat{\mathcal{J}}^{\mu}_a\rangle^{\mathrm{LG}}_{\bar{t}},\end{aligned} \tag{2.46}$$

where we used the conservation laws (2.29) and (2.33) for operators. We also used

$$\partial_{\bar{t}}\int d\Sigma_{\bar{t}\mu} f^{\mu} = \int d\Sigma_{\bar{t}} N\nabla_\mu f^\mu, \tag{2.47}$$

for an arbitrary smooth function $f^\mu(x)$ (see Appendix A.1). Here we introduced $\nabla_\mu\Lambda^a$ as

$$\nabla_\mu\Lambda^\nu \equiv \nabla_\mu\beta^\nu, \quad \nabla_\mu\Lambda^4 \equiv \nabla_\mu\nu + F_{\nu\mu}\beta^\nu \tag{2.48}$$

From Eq. (2.46), we obtain the divergence of ψ^μ as

$$\nabla_\mu\psi^\mu = (\nabla_\mu\Lambda^a)\langle \hat{\mathcal{J}}^{\mu}_a\rangle^{\mathrm{LG}}_{\bar{t}}. \tag{2.49}$$

If we write down it in a more explicit way, it gives

$$\begin{aligned}\partial_{\bar{t}}\Psi[\bar{t};\lambda] &= \int d\Sigma_{\bar{t}} N(\nabla_\mu\Lambda^a)\langle \hat{\mathcal{J}}^{\mu}_a\rangle^{\mathrm{LG}}_{\bar{t}} \\ &= \int d^{d-1}\bar{x}\sqrt{-g}\left(\nabla_\mu\beta_\nu\langle \hat{T}^{\mu\nu}\rangle^{\mathrm{LG}}_{\bar{t}} + (\nabla_\mu\nu + F_{\nu\mu}\beta^\nu)\langle \hat{J}^{\mu}\rangle^{\mathrm{LG}}_{\bar{t}}\right).\end{aligned} \tag{2.50}$$

To take one more step forward, we choose the useful coordinate system by matching the time vector $t^\mu(x) \equiv \partial_{\bar{t}}x^\mu(\bar{t},\bar{\boldsymbol{x}})$ with the local fluid vector $\beta^\mu(x)$: $t^\mu(x) = \beta^\mu(x)$. This gauge fixing is schematically shown in Fig. 2.3. Since the fluid remains at rest in this coordinate system, we call it the hydrostatic gauge[7]. Besides, we interpret the chemical potential as the time component of the background $U(1)$ gauge field: $\nu = A_\nu\beta^\nu = A_{\bar{0}}$. Under this parametrization, we obtain

$$\begin{aligned}\partial_{\bar{t}}\Psi[\bar{t};\lambda] &= \int d^{d-1}\bar{x}\sqrt{-g}\left(\nabla_\mu\beta_\nu\langle \hat{T}^{\mu\nu}\rangle^{\mathrm{LG}}_{\bar{t}} + (\nabla_\mu\nu + F_{\nu\mu}\beta^\nu)\langle \hat{J}^{\mu}\rangle^{\mathrm{LG}}_{\bar{t}}\right) \\ &= \int d^{d-1}\bar{x}\sqrt{-g}\left(\frac{1}{2}(\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu)\langle \hat{T}^{\mu\nu}\rangle^{\mathrm{LG}}_{\bar{t}} + (\beta^\nu\nabla_\nu A_\mu + A_\nu\nabla_\mu\beta^\nu)\langle \hat{J}^{\mu}\rangle^{\mathrm{LG}}_{\bar{t}}\right),\end{aligned} \tag{2.51}$$

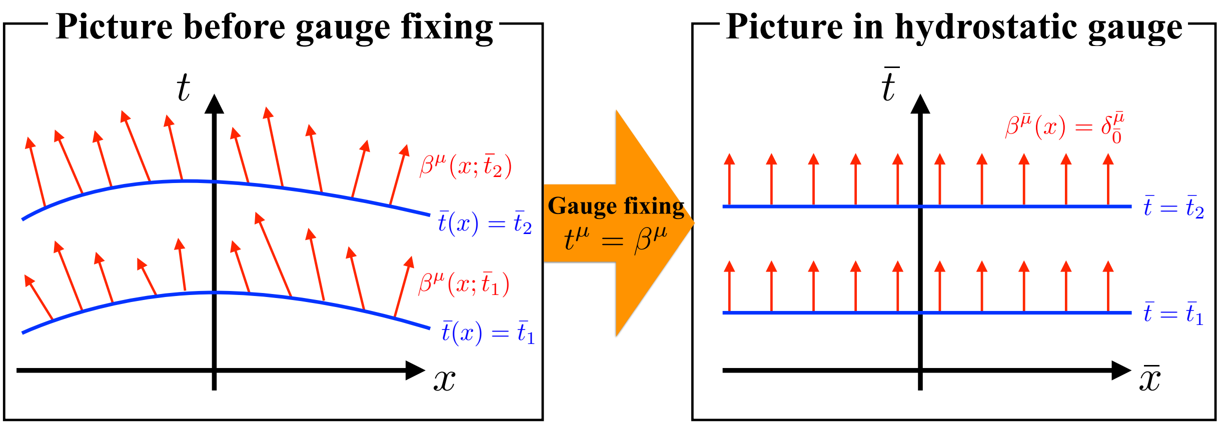


Figure 2.3: Schematic picture of a gauge fixing to the hydrostatic gauge. Before the gauge fixing, we have a fluid configuration $\beta^\mu(x)$ on the hypersurface, which could be any time-like vector. After the gauge fixing by choosing $t^\mu(x) = \beta^\mu(x)$, it simply becomes a set of the unit vectors.

where we used the symmetry of energy-momentum tensor under $\mu \leftrightarrow \nu$.

It closely resembles the variation formula of the action in the second line of Eq. (2.26). In fact, if we note that Lie derivatives of the metric and the external gauge field along the fluid-velocity β^μ are given by

$$\begin{aligned} \delta_\beta g_{\mu\nu} &= \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu, \\ \delta_\beta A_\mu &= \beta^\nu \nabla_\nu A_\mu + A_\nu \nabla_\mu \beta^\nu, \end{aligned} \tag{2.52}$$

we can rewrite the above equation as

$$\partial_{\bar{t}} \Psi[\bar{t};\lambda] = \int d^{d-1}\bar{x} \sqrt{-g} \left(\frac{1}{2} \delta_\beta g_{\mu\nu} \langle \hat{T}^{\mu\nu} \rangle^{\mathrm{LG}}_{\bar{t}} + \delta_\beta A_\mu \langle \hat{J}^\mu \rangle^{\mathrm{LG}}_{\bar{t}} \right). \tag{2.53}$$

On the other hand, the variation of the Massieu-Planck functional with respect to $\bar{t}$, or the Lie derivative along t^μ, is expressed in another way. Since we choose $t^\mu = \beta^\mu$, we can use the relation like $\delta_t \beta^\mu = \delta_\beta \beta^\mu = 0$. As a result, we obtain another expression

$$\partial_{\bar{t}} \Psi[\bar{t};\lambda] = \int d^{d-1}\bar{x} \left(\delta_\beta g_{\mu\nu} \frac{\delta \Psi}{\delta g_{\mu\nu}} + \delta_\beta A_\mu \frac{\delta \Psi}{\delta A_\mu} \right). \tag{2.54}$$

Comparison of Eq. (2.53) with Eq. (2.54) enables us to relate the average values of the conserved currents over the local Gibbs distribution with the variation of the Masseiu-Planck functional like

$$\langle \hat{T}^{\mu\nu}(x) \rangle^{\mathrm{LG}}_{\bar{t}} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \Psi[\bar{t};\lambda], \quad \langle \hat{J}^\mu(x) \rangle^{\mathrm{LG}}_{\bar{t}} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta A_\mu(x)} \Psi[\bar{t};\lambda]. \tag{2.55}$$

[7] The same name for a similar situation is also employed in Ref. [88].

In conclusion, we can identify the Masseiu-Planck functional as a generating functional for nondissipative hydrodynamics, in which we neglect the deviation from the local Gibbs distribution at each time.

2.3 Path integral formulation and emergent curved spacetime

In this section, dealing with some representative examples of quantum fields such as the scalar field, gauge field, and Dirac field, we explicitly perform path-integral analysis for the Masseiu-Planck functional $\Psi[\bar{t};\lambda]$ [1, 136]. As a result, we show that the Masseiu-Planck functional is written in terms of the Euclidean action in the same way as the case of the global thermal equilibrium. It, however, does not have the form of that in the flat spacetime, but in the emergent curved spacetime background, whose metric or vielbein is determined by the local temperature and the fluid four-velocity.

2.3.1 Scalar field

Real scalar field

Let us first consider a one-component real scalar field. In the coordinate system $(\bar{t},\bar{\boldsymbol{x}})$ with the ADM metric (2.16), the Lagrangian for a neutral scalar field ϕ reads

$$\mathcal{L} = -\frac{g^{\bar{\mu}\bar{\nu}}}{2}\partial_{\bar{\mu}}\phi\partial_{\bar{\nu}}\phi - V(\phi) = \frac{1}{2N^2}(\partial_{\bar{t}}\phi - N^{\bar{i}}\partial_{\bar{i}}\phi)^2 - \frac{\gamma^{\bar{i}\bar{j}}}{2}\partial_{\bar{i}}\phi\partial_{\bar{j}}\phi - V(\phi), \tag{2.56}$$

where $V(\phi)$ denotes the potential term. The canonical momentum $\pi(\boldsymbol{x})$ is $\pi(\boldsymbol{x}) \equiv -g^{\bar{0}\bar{\nu}}\partial_{\bar{\nu}}\phi(\bar{\boldsymbol{x}}) = N^{-2}(\partial_{\bar{t}}\phi - N^{\bar{i}}\partial_{\bar{i}}\phi)$, which satisfies the canonical commutation relation, $[\hat{\phi}(\bar{\boldsymbol{x}}),\hat{\pi}(\bar{\boldsymbol{x}}')] = i\delta(\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}')/(N\sqrt{\gamma})$. We obtain the energy-momentum tensors as

$$\hat{T}^{\bar{0}}_{\ \bar{0}} = -\hat{\pi}\partial_{\bar{t}}\hat{\phi} + \hat{\mathcal{L}} = -\frac{N^2}{2}\hat{\pi}^2 - N^{\bar{i}}\hat{\pi}\partial_{\bar{i}}\hat{\phi} - \frac{\gamma^{\bar{i}\bar{j}}}{2}\partial_{\bar{i}}\hat{\phi}\partial_{\bar{j}}\hat{\phi} - V(\hat{\phi}), \tag{2.57}$$

$$\hat{T}^{\bar{0}}_{\ \bar{i}} = -\hat{\pi}\partial_{\bar{i}}\hat{\phi}. \tag{2.58}$$

By using the standard technique of the path integral, we have

$$\begin{aligned}\operatorname{Tr} e^{-\hat{K}} &= \int d\phi\langle\phi|e^{-\hat{K}}|\phi\rangle \\ &= \int \mathcal{D}\phi\mathcal{D}\pi \exp\left(\int_0^{\beta_0} d\tau[i\int d^{d-1}\bar{x}N\sqrt{\gamma}\partial_\tau\phi(\tau,\bar{\boldsymbol{x}})\pi(\tau,\bar{\boldsymbol{x}}) - \beta_0^{-1}K]\right),\end{aligned} \tag{2.59}$$

where K denotes the functional corresponding to the operator $\hat{K}$. After parametrizing $\beta^{\bar{\mu}} = \beta_0 e^{\sigma}u^{\bar{\mu}}$ and integrating Eq. (2.59) with respect to the canonical momentum π, we obtain the path-integral formula for the Massieu-Planck functional as

$$\Psi[\bar{t};\lambda] = \log \int \mathcal{D}\phi \exp\left(S[\phi;\lambda]\right), \tag{2.60}$$

with

$$\begin{aligned} S[\phi;\lambda] &= \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x} \sqrt{\gamma}\tilde{N}\Big[\frac{1}{2\tilde{N}^2}\Big(i\partial_\tau\phi - \tilde{N}^{\bar{i}}\partial_{\bar{i}}\phi\Big)^2 - \Big(\frac{\gamma^{\bar{i}\bar{j}}}{2}\partial_{\bar{i}}\phi\partial_{\bar{j}}\phi + V(\phi)\Big)\Big] \\ &\equiv \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{-\tilde{g}}\tilde{\mathcal{L}}(\phi, \tilde{\partial}_{\bar{\rho}}\phi; \tilde{g}_{\bar{\mu}\bar{\nu}}), \end{aligned} \tag{2.61}$$

where $\tilde{N} \equiv N u^{\bar{0}} e^{\sigma} = -n_\mu \beta^\mu/\beta_0$, $\tilde{N}^{\bar{i}} \equiv \gamma^{\bar{i}\bar{j}} e^{\sigma} u_{\bar{j}} = e^{\sigma}(u^{\bar{0}} N^{\bar{i}} + u^{\bar{i}})$. Here we defined the partial derivative in the thermal space denoted by tilde such as $\tilde{\partial}_\mu \equiv (i\partial_\tau, \partial_{\bar{i}})$. We also defined the thermal metric $\tilde{g}_{\bar{\mu}\bar{\nu}}$ and its inverse $\tilde{g}^{\bar{\mu}\bar{\nu}}$ as

$$\tilde{g}_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} -\tilde{N}^2 + \tilde{N}_{\bar{i}}\tilde{N}^{\bar{i}} & \tilde{N}_{\bar{j}} \\ \tilde{N}_{\bar{i}} & \gamma_{\bar{i}\bar{j}} \end{pmatrix}, \qquad \tilde{g}^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} -\tilde{N}^{-2} & \tilde{N}^{-2}\tilde{N}^{\bar{j}} \\ \tilde{N}^{-2}\tilde{N}^{\bar{i}} & \gamma^{\bar{i}\bar{j}} - \tilde{N}^{-2}\tilde{N}^{\bar{i}}\tilde{N}^{\bar{j}} \end{pmatrix}. \tag{2.62}$$

Here, $\tilde{N}_{\bar{i}} \equiv \gamma_{\bar{i}\bar{j}}\tilde{N}^{\bar{j}} = e^{\sigma} u_{\bar{i}}$. As is clearly demonstrated from Eq. (2.60) to Eq. (2.62), the Masseiu-Planck functional $\Psi[\bar{t};\lambda]$ is expressed in terms of the path integral over the Euclidean action in the emergent curved spacetime, whose metric is given in Eq. (2.62).

Charged scalar field

We can easily generalize our analysis to a charged scalar field in a straightforward way. This system, however, is distinct from simple summation of two independent real field in a sense that there exists conserved charge curent coupled to the external gauge field, and thus can have chemical potential. Dealing with the charged scalar field, we show how the chemical potential and external gauge field is implemented in our path-integral formula.

Lagrangian for a charged scalar boson is given by

$$\begin{aligned} \mathcal{L} &= -g^{\bar{\mu}\bar{\nu}} D_{\bar{\mu}}\Phi^* D_{\bar{\nu}}\Phi - V(|\Phi|^2) \\ &= \frac{1}{N^2}(D_{\bar{t}}\Phi^* - N^{\bar{i}} D_{\bar{i}}\Phi^*)(D_{\bar{t}}\Phi - N^{\bar{i}} D_{\bar{i}}\Phi) - \gamma^{\bar{i}\bar{j}} D_{\bar{i}}\Phi^* D_{\bar{j}}\Phi - V(|\Phi|^2), \end{aligned} \tag{2.63}$$

where Φ denotes a complex field and describes bosons with positive and negative charges, and $D_{\bar{\mu}}$ is a covariant derivative which acts on charged fields as

$$D_{\bar{\mu}}\Phi = \partial_{\bar{\mu}}\Phi - iA_{\bar{\mu}}\Phi, \quad D_{\bar{\mu}}\Phi^* = \partial_{\bar{\mu}}\Phi^* + iA_{\bar{\mu}}\Phi^*, \tag{2.64}$$

where we take a charge of the complex field as unity, and $A_{\bar{\mu}}$ denotes an external gauge field coupled to the conserved charge current. Since this Lagrangian is invariant under $U(1)$ transformation $\Phi \to \Phi' = \Phi e^{i\alpha}$, this system possesses the conserved charge current,

$$J^{\bar{\mu}} = -ig^{\bar{\mu}\bar{\nu}}(\Phi^* D_{\bar{\nu}}\Phi - \Phi D_{\bar{\nu}}\Phi^*). \tag{2.65}$$

For the convenience, we decompose Φ into real and imaginary parts, $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$, in which both of ϕ_1 and ϕ_2 denote real fields, and rewrite the Lagrangian in the following form,

$$\begin{aligned}\mathcal{L} &= \sum_{a=1}^{2}\left[-\frac{g^{\bar\mu\bar\nu}}{2}D_{\bar\mu}\phi_a D_{\bar\nu}\phi_a\right] - V(\phi_1^2+\phi_2^2)\\ &= \sum_{a=1}^{2}\left[\frac{1}{2N^2}(D_{\bar t}\phi_a - N^{\bar i}D_{\bar i}\phi_a)^2 - \frac{\gamma^{\bar i\bar j}}{2}D_{\bar i}\phi_a D_{\bar j}\phi_a\right] - V(\phi_1^2+\phi_2^2).\end{aligned} \tag{2.66}$$

Here we introduced a covariant derivative, which acts to real fields ϕ_a $(a = 1, 2)$ as

$$D_{\bar\mu}\phi_a = \partial_{\bar\mu}\phi_a + \varepsilon_{ab}A_{\bar\mu}\phi_b \tag{2.67}$$

with $\varepsilon_{12} = 1 = -\varepsilon_{21},\ \varepsilon_{11} = \varepsilon_{22} = 0$, and use a contraction rule for the subscript b.

By using canonical momenta $\pi_a \equiv -g^{\bar0\bar\nu}D_{\bar\nu}\phi_a = N^{-2}(D_{\bar t}\phi_a - N^{\bar i}D_{\bar i}\phi_a)$ $(a = 1, 2)$, which satisfy the canonical commutation relations, $[\hat\phi_a(\bar{\boldsymbol{x}}), \hat\pi_b(\bar{\boldsymbol{x}}')] = i\delta_{ab}\delta(\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}')/(N\sqrt{\gamma})$, all the conserved charge densities such as the energy-momentum and conserved $U(1)$ charge are written as

$$\hat T^{\bar0}_{\ \bar0} = -\sum_{a=1}^{2}\left[\frac{N^2}{2}\hat\pi_a^2 + N^{\bar i}\hat\pi_a D_{\bar i}\hat\phi_a + \frac{\gamma^{\bar i\bar j}}{2}D_{\bar i}\hat\phi_a D_{\bar j}\hat\phi_a\right] - V(\hat\phi_1^2+\hat\phi_2^2), \tag{2.68}$$

$$\hat T^{\bar0}_{\ \bar i} = -\sum_{a=1}^{2}\hat\pi_a D_{\bar i}\hat\phi_a, \tag{2.69}$$

$$\hat J^{\bar0} = \sum_{a=1}^{2}\hat\pi_a\varepsilon_{ab}\hat\phi_b. \tag{2.70}$$

While these do not contain the time derivative, and thus the time component of the external gauge field $A_{\bar0}$, these are manifestly gauge invariant under the gauge transformation of the spatial components: $A_{\bar i} \to A_{\bar i} + \partial_{\bar i}\alpha$.

From this set of conserved quantities we obtain

$$\mathrm{Tr}\, e^{-\hat K} = \int \mathcal{D}\phi_1\mathcal{D}\phi_2\mathcal{D}\pi_1\mathcal{D}\pi_2 \exp\left(\int_0^{\beta_0} d\tau[i\int d^{d-1}\bar x N\sqrt{\gamma}\sum_{a=1}^{2}\partial_\tau\phi_a(\tau,\bar{\boldsymbol{x}})\pi_a(\tau,\bar{\boldsymbol{x}}) - \beta_0^{-1}K]\right). \tag{2.71}$$

Since the canonical momenta are quadratic, we are able to integrate out π_a $(a = 1, 2)$ also for this case. Under the same parametrization $\beta^{\bar\mu} = \beta_0 e^\sigma u^{\bar\mu}$, we can write down the Masseiu-Planck functional as

$$\Psi[\bar t; \lambda, A_{\bar\mu}] = \log\int \mathcal{D}\Phi \exp\left(S[\Phi; \lambda, A_{\bar\mu}]\right), \tag{2.72}$$

with

$$\begin{aligned} S[\Phi;\lambda,A_\mu] &= \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{\gamma}\tilde{N} \\ &\quad\times \left[\frac{1}{2\tilde{N}^2}\sum_{a=1}^{2}\left[\left(\tilde{D}_{\bar{0}}\phi_a - \tilde{N}^{\bar{i}}D_{\bar{i}}\phi_a\right)^2 - \frac{\gamma^{\bar{i}\bar{j}}}{2}D_{\bar{i}}\phi_a D_{\bar{j}}\phi_a\right] - V(|\Phi|^2)\right)\right] \\ &= \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{\gamma}\tilde{N} \\ &\quad\times \left[\frac{1}{\tilde{N}^2}\left(\tilde{D}_{\bar{0}}\Phi^* - \tilde{N}^{\bar{i}}\tilde{D}_{\bar{i}}\Phi^*\right)\left(\tilde{D}_{\bar{0}}\Phi - \tilde{N}^{\bar{i}}\tilde{D}_{\bar{i}}\Phi\right) - \gamma^{\bar{i}\bar{j}}\tilde{D}_{\bar{i}}\Phi^*\tilde{D}_{\bar{j}}\Phi - V(|\Phi|^2)\right] \\ &\equiv \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{-\tilde{g}}\tilde{\mathcal{L}}(\Phi,\tilde{\partial}_{\bar{\rho}}\Phi;\tilde{g}_{\bar{\mu}\bar{\nu}},\tilde{A}_{\bar{\mu}}), \end{aligned} \tag{2.73}$$

where we define the covariant derivative in the thermal space denoted by tilde as follows:

$$\tilde{D}_{\bar{\mu}}\phi_a = \tilde{\partial}_{\bar{\mu}}\phi_a + \varepsilon_{ab}\tilde{A}_{\bar{\mu}}\phi_b, \tag{2.74}$$

$$\tilde{D}_{\bar{\mu}}\Phi = \tilde{\partial}_{\bar{\mu}}\Phi - i\tilde{A}_{\bar{\mu}}\Phi, \quad \tilde{D}_{\bar{\mu}}\Phi^* = \tilde{\partial}_{\bar{\mu}}\Phi^* + i\tilde{A}_{\bar{\mu}}\Phi^*, \tag{2.75}$$

with the background gauge field defined by

$$\tilde{A}_{\bar{0}} \equiv e^\sigma\mu = \nu/\beta_0, \quad \text{and} \quad \tilde{A}_{\bar{i}} \equiv A_{\bar{i}}. \tag{2.76}$$

We note that $(i\partial_\tau)^\dagger = i\partial_\tau$ in our convention.

We see that the resulting Euclidean action is again written in terms of the thermal metric background (2.62), and an essential difference is only seen in the covariant derivative (2.74) or (2.75). We, therefore, only need to consider the modified gauge connection in the presence of finite chemical potential, by replacing the partial derivative $\tilde{\partial}_{\bar{\mu}}$ with the covariant one, $\tilde{D}_{\bar{\mu}}$. As is discussed in Sec. 2.4, this additional term $e^\sigma\mu = \nu/\beta_0$ is Kaluza-Klein gauge invariant. Therefore, the structure and symmetry properties of the emergent curved spacetime also hold for systems with finite chemical potential.

2.3.2 Gauge field

Abelian gauge field

As a next example, let us consider the electromagnetic field, whose field strength tensor is given by

$$F_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\mu}}A_{\bar{\nu}} - \partial_{\bar{\nu}}A_{\bar{\mu}}, \tag{2.77}$$

where $A_{\bar{\mu}}$ denotes the four-vector potential. The Lagrangian for the electromagnetic field is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}g^{\bar{\mu}\bar{\nu}}g^{\bar{\alpha}\bar{\beta}}F_{\bar{\mu}\bar{\alpha}}F_{\bar{\nu}\bar{\beta}}, \\ &= \frac{1}{2N^2}\gamma^{\bar{i}\bar{j}}(F_{\bar{0}\bar{i}} - N^{\bar{k}}F_{\bar{k}\bar{i}})(F_{\bar{0}\bar{j}} - N^{\bar{l}}F_{\bar{l}\bar{j}}) - \frac{1}{4}\gamma^{\bar{i}\bar{j}}\gamma^{\bar{k}\bar{l}}F_{\bar{i}\bar{k}}F_{\bar{j}\bar{l}}, \end{aligned} \tag{2.78}$$

where we use the coordinate system $(\bar{t}, \bar{\boldsymbol{x}})$ with the ADM metric (2.16) in the last line.

Since the field strength tensor is invariant under the gauge transformation

$$A_{\bar{\mu}}(x) \to A_{\bar{\mu}}(x) + \partial_{\bar{\mu}}\alpha(x), \tag{2.79}$$

where $\alpha(x)$ is an arbitrary function smoothly dependent on x, the Lagrangian and all physical observables are also gauge invariant. However, to quantize gauge field in our setup, which is essentially Hamiltonian formalism, we need to fix a gauge. Here, we employ the axial gauge

$$A_{\overline{d-1}}(x) = 0. \tag{2.80}$$

Although this axial gauge condition does not completely fix the gauge, we fix the residual gauge freedom later on.

The canonical momenta $\Pi^{\bar{i}}$ are given by

$$\Pi^{\bar{i}} \equiv -F^{\bar{0}\bar{i}} = \frac{1}{N^2}\gamma^{\bar{i}\bar{j}}(F_{\bar{0}\bar{j}} - N^{\bar{k}}F_{\bar{k}\bar{j}}). \tag{2.81}$$

Note that $\Pi^{\bar{0}} = 0$ so that $A_{\bar{0}}$ is not a dynamical field, since the field strength tensor is anti-symmetric under the exchange of indices. We also note that due to the axial gauge condition $A_{\overline{d-1}} = 0$, we do not have the $\Pi^{\overline{d-1}}$ as a dynamical field. In fact, it is determined by the Gauss's law

$$\nabla_{\bar{i}}F^{\bar{0}\bar{i}} = 0, \tag{2.82}$$

where we consider the situation in the absence of the charged particles.

From the Lagrangian for the electromagnetic field, we can construct energy-momentum tensor $\hat{T}^{\mu}{}_{\nu}$ as usual, and it gives

$$\hat{T}^{\bar{0}}{}_{\bar{0}} = \hat{F}^{\bar{0}\bar{\alpha}}\hat{F}_{\bar{0}\bar{\alpha}} + \hat{\mathcal{L}} = -\frac{N^2}{2}\hat{\Pi}^{\bar{i}}\gamma_{\bar{i}\bar{j}}\hat{\Pi}^{\bar{j}} - N^{\bar{i}}\hat{F}_{\bar{i}\bar{j}}\hat{\Pi}^{\bar{j}} - \frac{1}{4}\gamma^{\bar{i}\bar{j}}\gamma^{\bar{k}\bar{l}}\hat{F}_{\bar{i}\bar{k}}\hat{F}_{\bar{j}\bar{l}}, \tag{2.83}$$

$$\hat{T}^{\bar{0}}{}_{\bar{i}} = \hat{F}^{\bar{0}\bar{\alpha}}\hat{F}_{\bar{i}\bar{\alpha}} = -\hat{\Pi}^{\bar{j}}\hat{F}_{\bar{i}\bar{j}}. \tag{2.84}$$

As is mentioned before, contrary to its apparent expression, $\Pi^{\overline{d-1}}$ is not an independent dynamical field, and determined by solving Gauss's law (2.82) : $\Pi^{\overline{d-1}} = -F^{\bar{0}\overline{d-1}}(\Pi^{\bar{1}}, \cdots, \Pi^{\overline{d-2}})$. This fact is not useful in order to integrate out all the conjugate momentum $\Pi^{\bar{i}}$. Therefore, we insert an identity

$$1 = \int \mathcal{D}\Pi^{\overline{d-1}}\delta\left(\Pi^{\overline{d-1}} + F^{\bar{0}\overline{d-1}}(\Pi^{\bar{1}}, \cdots, \Pi^{\overline{d-2}})\right) \tag{2.85}$$

to avoid this apparent difficulty. Furthermore, by decomposing the Gauss law constraint as

$$\delta(\Pi^{\overline{d-1}} + F^{\bar{0}\overline{d-1}}(\Pi^{\bar{1}}, \cdots, \Pi^{\overline{d-2}})) = \delta(\nabla_{\bar{i}}\Pi^{\bar{i}})\det\left(\frac{\partial(\nabla_{\bar{i}}\Pi^{\bar{i}})}{\partial\Pi^{\overline{d-1}}}\right) = \delta(\nabla_{\bar{i}}\Pi^{\bar{i}})\det\left(\nabla_{\overline{d-1}}\right), \tag{2.86}$$

we express the partition function with the following path-integral formula

$$\begin{aligned}\operatorname{Tr} e^{-\hat{K}} &= \int \prod_{j=1}^{d-1} \mathcal{D}\Pi^{\bar{j}} \prod_{k=0}^{d-2} \mathcal{D}A_{\bar{k}}\, \delta(\nabla_{\bar{i}}\Pi^{\bar{i}}) \det(\nabla_{\overline{d-1}}) \\ &\quad \times \exp\left(\int_0^{\beta_0} d\tau \left[\int d^{d-1}\bar{x} N\sqrt{\gamma}\left(\sum_{l=1}^{d-2}\Pi^{\bar{l}} i\partial_\tau A_{\bar{l}}\right) - \beta_0^{-1}K\right]\right) \\ &= \int \prod_{j=1}^{d-1} \mathcal{D}\Pi^{\bar{j}} \prod_{k=0}^{d-2} \mathcal{D}A_{\bar{k}}\, \det(\nabla_{\overline{d-1}}) \\ &\quad \times \exp\left(\int_0^{\beta_0} d\tau \left[\int d^{d-1}\bar{x} N\sqrt{\gamma}\left(\sum_{l=1}^{d-2}\Pi^{\bar{l}} i\partial_\tau A_{\bar{l}} - \Pi^{\bar{i}} i\partial_{\bar{i}} A_{\bar{0}}\right) - \beta_0^{-1}K\right]\right),\end{aligned} \tag{2.87}$$

where we use a functional-integral expression for the delta function $\delta(\nabla_{\bar{i}}\Pi^{\bar{i}})$ with an auxiliary field $A_{\bar{0}}$

$$\delta(\nabla_{\bar{i}}\Pi^{\bar{i}}) = \int \mathcal{D}A_{\bar{0}} \exp\left(i\int_0^{\beta_0} d\tau \int d^{d-1}\bar{x} N\sqrt{\gamma} A_{\bar{0}} \nabla_{\bar{i}}\Pi^{\bar{i}}\right), \tag{2.88}$$

and perform an integration by parts in order to obtain the last line in Eq. (2.87).

Using the same parametrization $\beta^{\bar{\mu}} = \beta_0 e^{\sigma} u^{\bar{\mu}}$ with the case of the scalar fields, and after integrating out the conjugate momenta $\Pi^{\bar{i}}$, we obtain the path-integral formula for the Masseiu-Planck functional,

$$\Psi[\bar{t};\lambda] = \log \int \prod_{i=0}^{d-2} \mathcal{D}A_{\bar{i}} \det(\nabla_{\overline{d-1}}) \exp\left(S[A_{\bar{\sigma}};\lambda]\right), \tag{2.89}$$

with

$$\begin{aligned} S[A_{\bar{\sigma}};\lambda] &= \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{\gamma}\tilde{N} \\ &\quad \times \left[\frac{1}{2\tilde{N}^2}\gamma^{\bar{i}\bar{j}}(\tilde{F}_{\bar{0}\bar{i}} - \tilde{N}^{\bar{k}}F_{\bar{k}\bar{i}})(\tilde{F}_{\bar{0}\bar{j}} - \tilde{N}^{\bar{l}}F_{\bar{l}\bar{j}}) - \frac{1}{4}\gamma^{\bar{i}\bar{j}}\gamma^{\bar{k}\bar{l}}F_{\bar{i}\bar{k}}F_{\bar{j}\bar{l}}\right], \\ &\equiv \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{-\tilde{g}}\tilde{\mathcal{L}}(\tilde{\partial}_{\bar{\rho}}A_{\bar{\sigma}};\tilde{g}_{\bar{\mu}\bar{\nu}}), \end{aligned} \tag{2.90}$$

where $\tilde{N} \equiv Nu^{\bar{0}}e^{\sigma} = -n_\mu\beta^\mu/\beta_0$, $\tilde{N}^{\bar{i}} \equiv \gamma^{\bar{i}\bar{j}}e^{\sigma}u_{\bar{j}}$ is used as is the same with the case for the scalar fields, and we introduced the field strength tensor along the imaginary-time direction:

$$\tilde{F}_{\bar{0}\bar{i}} \equiv i\partial_\tau A_{\bar{i}} - i\partial_{\bar{i}}A_{\bar{0}}. \tag{2.91}$$

What is important is that the result is again written in terms of the Euclidean action under the curved spacetime with thermal metric (2.62).

A short comment on the gauge invariance is in order here. The above result is the path-integral formula of the Massieu-Planck functional for the axial gauge, and the path integral over $A_{\overline{d-1}}$ is not contained because of the axial gauge condition $A_{\overline{d-1}} = 0$. However, we can implement the axial gauge condition through an insertion of

$$1 = \int \mathcal{D}A_{\overline{d-1}}\delta(A_{\overline{d-1}}), \tag{2.92}$$

and, as a result, we obtain

$$\Psi[\bar{t};\lambda] = \log \int \mathcal{D}A_{\bar{\mu}}\delta(A_{\overline{d-1}}) \det(\nabla_{\overline{d-1}}) e^{S[A_{\bar{\sigma}};\lambda]}. \tag{2.93}$$

This is the result for a special choice of the axial gauge, but we can easily generalize this result for an arbitrary gauge choice by replacing the gauge fixing condition and Jacobian as

$$\delta(A_{\overline{d-1}}) \det(\nabla_{\overline{d-1}}) \to \delta(F) \det\left(\frac{\partial F}{\partial \alpha}\right), \tag{2.94}$$

where $F = 0$ gives the gauge fixing condition like $F = A_{\overline{d-1}}$ in the axial gauge. Since the delta function and the determinant give a gauge-invariant combination, the final expression for the Masseiu-Planck functional is

$$\Psi[\bar{t};\lambda] = \log \int \mathcal{D}A_{\bar{\mu}}\delta(F) \det\left(\frac{\partial F}{\partial \alpha}\right) \exp\left(S[A_{\bar{\sigma}};\lambda]\right), \tag{2.95}$$

which is explicitly gauge invariant so that we can choose an arbitrary gauge suitable for our calculation.

Non-Abelian gauge field

Let us generalize our result to the non-Abelian gauge field. Here, for concreteness, we consider SU(N) gauge theory. The Lagrangian for the non-abelian gauge field is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4} g^{\bar{\mu}\bar{\nu}} g^{\bar{\alpha}\bar{\beta}} G^a{}_{\bar{\mu}\bar{\alpha}} G^a{}_{\bar{\nu}\bar{\beta}} \\ &= \frac{1}{2N^2}\gamma^{\bar{i}\bar{j}}(G^a{}_{\bar{0}\bar{i}} - N^{\bar{k}} G^a{}_{\bar{k}\bar{i}})(G^a{}_{\bar{0}\bar{j}} - N^{\bar{l}} G^a{}_{\bar{l}\bar{j}}) - \frac{1}{4}\gamma^{\bar{i}\bar{j}}\gamma^{\bar{k}\bar{l}} j G^a{}_{\bar{i}\bar{k}} G^a{}_{\bar{j}\bar{l}}.\end{aligned} \tag{2.96}$$

Here we introduced the field strength tensor for the non-Abelian gauge field

$$G^a{}_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\mu}} A^a{}_{\bar{\nu}} - \partial_{\bar{\nu}} A^a{}_{\bar{\mu}} + g f_{abc} A^b{}_{\bar{\mu}} A^c{}_{\bar{\nu}}, \tag{2.97}$$

with the non-Abelian gauge field $A^a{}_{\bar{\mu}}$, the dimensionless coupling constant g, and the structure constants of $SU(N)$ gauge group f_{abc}, which satisfy

$$[t^a, t^b] = i f_{abc}, \quad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta_{ab}, \tag{2.98}$$

where t^a denotes generators of SU(N) group. One important difference with the Abelian gauge field is that the gauge field carries the (color) index a which runs from $a = 1$ to $N^2 - 1$.

Introducing $A_{\bar{\mu}} = t^a A^a{}_{\bar{\mu}}$, we can express the field strength tensor in terms of the commutator of the covariant derivative

$$G_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\mu}} A_{\bar{\nu}} - \partial_{\bar{\nu}} A_{\bar{\mu}} - ig[A_{\bar{\mu}}, A_{\bar{\nu}}] = \frac{i}{g}[D_{\bar{\mu}}, D_{\bar{\nu}}], \tag{2.99}$$

where we introduced $G_{\bar{\mu}\bar{\nu}} \equiv t^a G^a{}_{\bar{\mu}\bar{\nu}}$ and covariant derivative

$$D_{\bar{\mu}} \equiv \partial_{\bar{\mu}} - igt^a A^a{}_{\bar{\mu}}, \tag{2.100}$$

The field strength tensor transforms as $G_{\bar{\mu}\bar{\nu}} \to UG_{\bar{\mu}\bar{\nu}}U^\dagger$ under the $SU(N)$ gauge transformation

$$A_{\bar{\mu}}(x) \to U(x)(A_{\bar{\mu}}(x) + ig^{-1}\partial_{\bar{\mu}})U^\dagger(x), \tag{2.101}$$

where $U(x) \equiv \exp(-i\theta^a(x)t^a)$ is a unitary matrix: $UU^\dagger = \mathbb{1}$. Together with the cyclic property of traces: $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$, we easily see gauge invariance of the Lagrangian (2.96).

Quantization procedure of the non-Abelian gauge field is essentially in a similar way with the Abelian gauge fields, and we directly write down the final result for the Masseiu-Planck functional,

$$\Psi[\bar{t}; \lambda] = \log \int \mathcal{D}A^a{}_{\bar{\mu}}\delta(F^b) \det\left(\frac{\partial F^c}{\partial \alpha_d}\right) \exp\left(S[A_{\bar{\sigma}}; \lambda]\right), \tag{2.102}$$

where $\delta(F^b)$ represents the gauge-fixing condition, and the determinant does the Fadeev-Popov determinant with the gauge parameter α^d. The Euclidean action is completely the same as the previous analysis on the Abelian case and

$$\begin{aligned} S[A_{\bar{\sigma}}; \lambda] = &\int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{\gamma}\tilde{N} \\ &\times \left[\frac{1}{2\tilde{N}^2}\gamma^{\bar{i}\bar{j}}(\tilde{G}^a{}_{\bar{0}\bar{i}} - \tilde{N}^{\bar{k}}G^a{}_{\bar{k}\bar{i}})(\tilde{G}^a{}_{\bar{0}\bar{j}} - \tilde{N}^{\bar{l}}G^a{}_{\bar{l}\bar{j}}) - \frac{1}{4}\gamma^{\bar{i}\bar{j}}\gamma^{\bar{k}\bar{l}}G^a{}_{\bar{i}\bar{k}}G^a{}_{\bar{j}\bar{l}}\right], \\ \equiv &\int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\sqrt{-\tilde{g}}\tilde{\mathcal{L}}(\tilde{\partial}_{\bar{\rho}}A_{\bar{\sigma}}; \tilde{g}_{\bar{\mu}\bar{\nu}}). \end{aligned} \tag{2.103}$$

Here in the same way as the Abelian case, we introduced the field strength tensor along the imaginary-time direction:

$$\tilde{G}^a{}_{\bar{0}\bar{i}} \equiv i\partial_\tau A^a{}_{\bar{i}} - i\partial_{\bar{i}} A^a{}_{\bar{0}}. \tag{2.104}$$

2.3.3 Dirac field

Spinor field in curved spacetime

As a last example, let us consider the Dirac field. Before starting the path-integral analysis in the case of the Dirac field, we first summarize a way to describe spinor fields in the curved spacetime, that is, a so-called vielbein formalism.

In order to describe the spinor field in the curved spacetime, we use the vielbein ${e_\mu}^a$ instead of the metric $g_{\mu\nu}$. Here, Greek letters $(\mu, \nu, \cdots)$ represent the curved spacetime indices in the coordinate system $(t, \boldsymbol{x})$, while Latin letters $(a, b, \cdots)$ do the local Lorentz indices. The metric and vielbein are related to each other through

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab}, \quad \eta^{ab} = {e_\mu}^a {e_\nu}^b g^{\mu\nu}. \tag{2.105}$$

We also define the inverse vielbein ${e_a}^\mu$, which satisfies the relations $\delta_\mu^\nu = {e_\mu}^a {e_a}^\nu$, $\delta_a^b = {e_a}^\mu {e_\mu}^b$. The (inverse) vielbein enables us to exchange the curved spacetime indices and the local Lorentz indices as follows

$$\begin{aligned} B_a &= {e_a}^\mu B_\mu, \quad B^a = {e_\mu}^a B^\mu, \\ B_\mu &= {e_\mu}^a B_a, \quad B^\mu = {e_a}^\mu B^a. \end{aligned} \tag{2.106}$$

The Lagrangian for a Dirac field ψ is expressed by the use of the inverse vielbein

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}({e_a}^\mu \gamma^a \overrightarrow{D}_\mu - \overleftarrow{D}_\mu \gamma^a {e_a}^\mu)\psi - m\bar{\psi}\psi, \tag{2.107}$$

where we defined $\bar{\psi} \equiv i\psi^\dagger \gamma^0$, and the covariant derivative

$$D_\mu = \partial_\mu - iA_\mu - i\frac{1}{2}{\omega_\mu}^{ab}\Sigma_{ab}, \tag{2.108}$$

with the external gauge field A_μ, and a spin connection ${\omega_\mu}^{ab}$. Here $\Sigma_{ab} \equiv i[\gamma_a, \gamma_b]/4$ is a generator of the Lorentz group with γ^a being the gamma matrices, which satisfy a set of relations $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ with $\{A, B\} \equiv AB + BA$, $(i\gamma^0)^\dagger = i\gamma^0$, $(i\gamma^0)^\dagger(i\gamma^0) = 1$, $i\gamma^0(\gamma^a)^\dagger i\gamma^0 = -\gamma^a$, and $i\gamma^0 \Sigma_{ab}^\dagger i\gamma^0 = \Sigma_{ab}$ in our convention. From a direct calculation, we obtain the next relations

$$[\Sigma_{ab}, \Sigma_{cd}] = -i(\eta_{ac}\Sigma_{bd} - \eta_{bc}\Sigma_{ad} - \eta_{ad}\Sigma_{bc} + \eta_{bd}\Sigma_{ac}), \tag{2.109}$$

$$[\gamma_a, \Sigma_{cd}] = -i(\eta_{ad}\gamma_c - \eta_{ac}\gamma_d). \tag{2.110}$$

The left and right derivatives are defined as

$$\overrightarrow{D}_\mu \phi \equiv \partial_\mu \phi - iA_\mu \phi - i\frac{1}{2}{\omega_\mu}^{ab}\Sigma_{ab}\phi, \quad \text{and} \quad \phi\overleftarrow{D}_\mu \equiv \partial_\mu \phi + i\phi A_\mu + i\phi\frac{1}{2}{\omega_\mu}^{ab}\Sigma_{ab}. \tag{2.111}$$

In Eq. (2.108), we have the spin connection ${\omega_\mu}^{ab}$, which is expressed by the vielbein as

$$\begin{aligned} {\omega_\mu}^{ab} &= \frac{1}{2}e^{a\nu}e^{b\rho}(C_{\nu\rho\mu} - C_{\rho\nu\mu} - C_{\mu\nu\rho}), \\ C_{\mu\nu\rho} &\equiv {e_\mu}^c(\partial_\nu e_{\rho c} - \partial_\rho e_{\nu c}), \end{aligned} \tag{2.112}$$

where $C_{\mu\nu\rho}$ are called the Ricci rotation coefficients. We note that the spin connection ${\omega_\mu}^{ab}$ is anti-symmetric under the exchange of the local Lorentz indices: ${\omega_\mu}^{ab} = -{\omega_\mu}^{ba}$.

Energy-momentum conservation for spinor field

As is demonstrated in Sec. 2.2.2, taking the variation of the action with respect to the metric, we obtain the conserved energy-momentum tensor associated with diffeomorphism invariance. However, if matters considered are composed of spinor fields, the action is described not by the metric $g_{\mu\nu}$ but by the vielbein $e_\mu^{\ a}$

$$S[\psi,\bar{\psi};e_\mu^{\ a},A_\mu]=\int d^dxe\mathcal{L}(\psi(x),\bar{\psi}(x),D_\mu\psi(x),D_\mu\bar{\psi}(x);e_\mu^{\ a}(x),A_\mu(x)), \tag{2.113}$$

where we define $e\equiv\det e_\mu^{\ a}=\sqrt{-g}$, and the explicit form of the Lagrangian for the Dirac field is already given by Eq. (2.107). In a similar way discussed in Sec. 2.2.2, we can generalize our discussion on the derivation of the energy-momentum conservation law for the fermionic action. Let us consider a set of variations with respect to the vielbein $e_\mu^{\ a}$, the external gauge field A_μ, and the spinor fields ψ, $\bar{\psi}$

$$\delta_\xi e_\mu^{\ a}\equiv e'^{\ a}_{\mu}(x)-e_\mu^{\ a}(x)=\xi^\nu\nabla_\nu e_\mu^{\ a}+e_\nu^{\ a}\nabla_\mu\xi^\nu, \tag{2.114}$$
$$\delta_\xi A_\mu\equiv A'_\mu(x)-A_\mu(x)=\xi^\nu\nabla_\nu A_\mu+A_\nu\nabla_\mu\xi^\nu, \tag{2.115}$$
$$\delta_\xi\psi\equiv\psi'(x)-\psi(x)=\xi^\nu\partial_\nu\psi, \tag{2.116}$$
$$\delta_\xi\bar{\psi}\equiv\bar{\psi}'(x)-\bar{\psi}(x)=\xi^\nu\partial_\nu\bar{\psi}, \tag{2.117}$$

which are caused by the general coordinate transformation (2.22). Since the action again has diffeomorphism invariance, the variation of the action under this transformations vanishes: $\delta S=0$. Since the variations of the fields does not contribute with the help of the equation of motion, the variation of the action leads to

$$\begin{aligned}\delta S&=\int d^dxe\left[\mathcal{T}^\mu_{\ a}\delta_\xi e_\mu^{\ a}+J^\mu\delta_\xi A_\mu\right]\\&=\int d^dxe\left[\mathcal{T}^\mu_{\ a}(\xi^\nu\nabla_\nu e_\mu^{\ a}+e_\nu^{\ a}\nabla_\mu\xi^\nu)+J^\mu(\xi^\nu\nabla_\nu A_\mu+A_\nu\nabla_\mu\xi^\nu)\right]\\&=-\int d^dxe\left[(\nabla_\mu\mathcal{T}^\mu_{\ \nu}-F_{\nu\lambda}J^\lambda)\xi^\nu\right]+\int d^dxe\mathcal{T}_{ab}\omega_\nu^{\ ab}\xi^\nu\\&\quad+\int d^dxe\nabla_\mu[(\mathcal{T}^\mu_{\ \nu}+J^\mu A_\nu)\xi^\nu]-\int d^dxe\xi^\nu A_\nu\nabla_\mu J^\mu,\end{aligned} \tag{2.118}$$

where we define the energy-momentum tensor for spinor fields $\mathcal{T}^\mu_{\ a}$ as

$$\mathcal{T}^\mu_{\ a}\equiv\frac{1}{e}\frac{\delta S}{\delta e_\mu^{\ a}}, \tag{2.119}$$

and we replace the Lorentz indices as the curved spacetime indices by the use of the vielbein: $\mathcal{T}^\mu_{\ \nu}=e_\nu^{\ a}\mathcal{T}^\mu_{\ a}$. Here we also used a so-called tetrad postulate that argues that the covariant derivative of the vielbein vanishes

$$D_\mu e_\nu^{\ b}=\nabla_\mu e_\nu^{\ b}+\omega_\mu^{\ b}{}_a e_\nu^{\ a}=0, \tag{2.120}$$

where $\Gamma^{\mu}{}_{\rho\sigma}$ denotes the usual Christoffel symbol without a torsion. Compared to the previous case in Eq. (2.26), we have the additional term proportional to $\mathcal{T}_{ab}\omega_{\mu}{}^{ab}$, which, in general, does not seem to vanish. However, as will be shown soon, this term vanishes, and we have the energy-momentum conservation law

$$\nabla_{\mu}\mathcal{T}^{\mu}_{\nu} = F_{\nu\lambda}J^{\lambda}. \tag{2.121}$$

Let us focus on the reason that the additional term does not contribute. In addition to diffeomorphism invariance, we have another symmetry due to the fact that it does not matter which locally inertial frames we adopt. In other words, the fermionic action is invariant under the local Lorentz transformation

$$\delta_{\alpha}e_{\mu}{}^{a} = \alpha^{a}{}_{b}(x)e_{\mu}{}^{b}, \tag{2.122}$$

$$\delta_{\alpha}\psi = -\frac{i}{2}\alpha^{ab}(x)\Sigma_{ab}\psi, \tag{2.123}$$

$$\delta_{\alpha}\bar{\psi} = \frac{i}{2}\alpha^{ab}(x)\Sigma_{ab}\bar{\psi}, \tag{2.124}$$

where $\alpha^{a}{}_{b}(x)$ denotes a local rotation angle, which is anti-symmetric: $\alpha_{ab}(x) = -\alpha_{ba}(x)$, and Σ_{ab} the generator of the Lorentz group. By the use of the equation of motion $\delta S/\delta\psi = \delta S/\delta\bar{\psi} = 0$, the variation of action under the infinitesimal local Lorentz transformation is, then, expressed as

$$\begin{aligned}\delta S &= \int d^{d}xe\mathcal{T}^{\mu}_{a}\delta_{\alpha}e_{\mu}{}^{a} = -\int d^{d}xe\mathcal{T}^{ab}\alpha_{ab} \\ &= -\int d^{d}xe\frac{1}{2}(\mathcal{T}^{ab} - \mathcal{T}^{ba})\alpha_{ab},\end{aligned} \tag{2.125}$$

for arbitrary $\alpha_{ab}(x)$. Therefore, local Lorentz invariance of the action: $\delta S = 0$, results in the proposition that the anti-symmetric part of the energy-momentum tensor vanishes:

$$\mathcal{T}^{ab} - \mathcal{T}^{ba} = 0. \tag{2.126}$$

This is why we drop the term proportional to $\mathcal{T}_{ab}\omega_{\mu}{}^{ab}$ in Eq. (2.118).

Combined with the consequence of differmorphism invariance and that of local Lorentz invariance, in other words, the energy-momentum conservation law (2.121), and the symmetry property of the energy-momentum tensor (2.126), we immediately conclude that the symmetric energy-momentum tensor is also conserved for the fermionic case,

$$\boxed{\nabla_{\mu}T^{\mu}_{\nu} = F_{\nu\lambda}J^{\lambda},} \tag{2.127}$$

where we define the symmetric energy-momentum tensor T^{μ}

$$T^{\mu\nu} = \frac{1}{2}(\mathcal{T}^{\mu}_{a}e^{\nu a} + \mathcal{T}^{\nu}_{a}e^{\mu a}), \tag{2.128}$$

which is clearly symmetric under $\mu \leftrightarrow \nu$ by definition.

Charge current conservation

The Lagrangian for the Dirac field (2.107) also has a gauge symmetry under the $U(1)$ gauge transformation: $\delta_\alpha A_\mu = \partial_\mu \alpha$. We, therefore, have a conserved vector current J^μ which is coupled to the background $U(1)$ gauge field:

$$\nabla_\mu J^\mu = 0, \quad \text{with} \quad J^\mu \equiv i\bar{\psi}\gamma^\mu\psi. \tag{2.129}$$

Path-integral formula for Dirac field

We are ready to develop the path-integral formulation for the Dirac field. First of all, taking the variation of the action with respect to vielbein, we obtain the energy-momentum tensor $\mathcal{T}^{\mu\nu}$ defined in Eq. (2.119) as

$$\mathcal{T}^{\mu\nu} = \frac{1}{2}\bar{\psi}(\gamma^\mu \overrightarrow{D}^\nu - \overleftarrow{D}^\nu \gamma^\mu)\psi - \frac{i}{4}D_\rho\big(\bar{\psi}\{\gamma^\mu, \Sigma^{\nu\rho}\}\psi\big) + g^{\mu\nu}\mathcal{L}. \tag{2.130}$$

By symmetrizing the indices, we also have the symmetric energy-momentum tensor $T^{\mu\nu}$ defined in Eq. (2.128)

$$T^{\mu\nu} \equiv \frac{1}{2}(\mathcal{T}^{\mu\nu} + \mathcal{T}^{\nu\mu}) = \frac{1}{4}\bar{\psi}(\gamma^\mu \overrightarrow{D}^\nu + \gamma^\nu \overrightarrow{D}^\mu - \overleftarrow{D}^\nu\gamma^\mu - \overleftarrow{D}^\mu\gamma^\nu)\psi + g^{\mu\nu}\mathcal{L}. \tag{2.131}$$

We have to choose which energy-momentum tensor we adopt in order to construct the local Gibbs distribution. Our choice is the symmetric energy-momentum tensor (2.131). The reason for this choice is as follows: Our guiding principal to construct the local Gibbs distribution is that we should collect a set of independent conserved quantities such as the energy, momentum, and conserved charge. We do not have to take into account the angular momentum as a conserved charge since if the energy-momentum tensor is symmetric, the associated angular momentum is trivially conserved, and hence, it is not the independent conserved quantity. One can also argue that the energy-momentum tensor appeared in relativistic hydrodynamics should be symmetric.

If we adopt the symmetric energy-momentum tensor, we have

$$\operatorname{Tr} e^{-\hat{K}} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(\int_0^{\beta_0} d\tau [i \int d^{d-1}\bar{x} e \frac{-1}{2}(\bar{\psi}\gamma^{\bar{0}}\overrightarrow{\partial_\tau}\psi - \bar{\psi}\overleftarrow{\partial_\tau}\gamma^{\bar{0}}\psi) - \beta_0^{-1}K]\right), \tag{2.132}$$

where K includes the symmetric energy-momentum tensor. Here we note that the imaginary-time derivative is not the covariant derivative but the partial derivative, because it simply arises from inner products of the adjacent state vectors introduced by the insertion of complete sets. On the other hand, the spatial derivative is the covariant derivative.

Contrary to the previous examples, we face with the problematic situation that the symmetric energy-momentum tensor does not seem to reproduce the correct Euclidean action. It is also not reasonable that the imaginary-time derivative is not covariant one, if the Euclidean action is given as that of the emergent curved spacetime. As will be shown below, these difficulties are closely related with each other, and a proper treatment again gives the correct Euclidean action in the emergent thermal spacetime.

In order to decompose the symmetric energy-momentum tensor, we use the consequence of local Lorentz invariance

$$\mathcal{T}^{ab} - \mathcal{T}^{ba} = 0 \Leftrightarrow \frac{1}{4}\bar{\psi}(\gamma^\mu \overrightarrow{D}_\nu - \gamma_\nu \overrightarrow{D}^\mu - \overleftarrow{D}_\nu \gamma^\mu + \overleftarrow{D}^\mu \gamma_\nu)\psi - \frac{i}{4} D_\rho(\bar{\psi}\{\gamma^\mu, \Sigma_\nu{}^\rho\}\psi) = 0. \tag{2.133}$$

By the virtue of this relation, we can rewrite the symmetric energy-momentum tensor as

$$\begin{aligned} T^\mu_{\ \nu} &= \frac{1}{2}\bar{\psi}(\gamma^\mu \overrightarrow{D}_\nu - \overleftarrow{D}_\nu \gamma^\mu)\psi + \delta^\mu_\nu \mathcal{L} - \frac{1}{4}\bar{\psi}(\gamma^\mu \overrightarrow{D}_\nu - \gamma_\nu \overrightarrow{D}^\mu - \overleftarrow{D}_\nu \gamma^\mu + \overleftarrow{D}^\mu \gamma_\nu)\psi \\ &= \frac{1}{2}\bar{\psi}(\gamma^\mu \overrightarrow{D}_\nu - \overleftarrow{D}_\nu \gamma^\mu)\psi + \delta^\mu_\nu \mathcal{L} - \frac{i}{4} D_\rho(\bar{\psi}\{\gamma^\mu, \Sigma_\nu{}^\rho\}\psi) \\ &\equiv \Theta^\mu{}_\nu - \frac{i}{2} D_\rho \Sigma^\mu{}_\nu{}^\rho, \end{aligned} \tag{2.134}$$

where we defined the canonical part of the energy-momentum tensor $\Theta^\mu{}_\nu$, and the spin part of the angular momentum tensor $\Sigma^\mu{}_\nu{}^\rho$ as

$$\Theta^\mu{}_\nu \equiv \frac{1}{2}\bar{\psi}(\gamma^\mu \overrightarrow{D}_\nu - \overleftarrow{D}_\nu \gamma^\mu)\psi + \delta^\mu_\nu \mathcal{L}, \tag{2.135}$$

$$\Sigma^\mu{}_\nu{}^\rho \equiv \frac{1}{2}(\bar{\psi}\{\gamma^\mu, \Sigma_\nu{}^\rho\}\psi). \tag{2.136}$$

Then, we can rewrite the K as follows:

$$\begin{aligned} K &= -\int d\Sigma_{\bar{t}\bar{\mu}} \left(\beta^{\bar{\nu}} T^{\bar{\mu}}_{\ \bar{\nu}} + \nu J^{\bar{\mu}}\right) \\ &= -\int d\Sigma_{\bar{t}\bar{\mu}} \left(\beta^{\bar{\nu}} \Theta^{\bar{\mu}}{}_{\bar{\nu}} - \frac{i}{2}\beta^{\bar{\nu}} D_{\bar{\rho}} \Sigma^{\bar{\mu}}{}_{\bar{\nu}}{}^{\bar{\rho}} + \nu J^{\bar{\mu}}\right) \\ &= -\int d\Sigma_{\bar{t}\bar{\mu}} \left(\beta^{\bar{\nu}} \Theta^{\bar{\mu}}{}_{\bar{\nu}} + \frac{i}{2}\Sigma^{\bar{\mu}}{}_{\bar{\nu}}{}^{\bar{\rho}} D_{\bar{\rho}} \beta^{\bar{\nu}} + \nu J^{\bar{\mu}}\right) + \frac{i}{2}\int dS_{\bar{t}\bar{\mu}\bar{\rho}} \Sigma^{\bar{\mu}}{}_{\bar{\nu}}{}^{\bar{\rho}} \beta^{\bar{\nu}}, \end{aligned} \tag{2.137}$$

where $dS_{\bar{t}\bar{\rho}\bar{\mu}}$ denotes the surface element for the $(d-1)-$dimensional spatial region $\Sigma_{\bar{t}}$, and we used the Stokes's theorem

$$\int_\Sigma d\Sigma_{\bar{t}\bar{\mu}} D_{\bar{\rho}} B^{\bar{\rho}\bar{\mu}} = \int_{\partial\Sigma} dS_{\bar{t}\bar{\mu}\bar{\rho}} B^{\bar{\rho}\bar{\mu}}, \tag{2.138}$$

satisfied for anti-symmetric tensors $B^{\bar{\mu}\bar{\nu}} = -B^{\bar{\nu}\bar{\mu}}$ to obtain the last line in Eq. (2.137). If the fields fall off sufficiently rapidly as $|\bar{\boldsymbol{x}}| \to \infty$, we can neglect the surface term.

Through the careful analysis we finally get the following expression:

$$\Psi[\bar{t}; \lambda, A_{\bar{\mu}}] = \log \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, \exp\left(S[\psi, \bar{\psi}; \lambda, A_{\bar{\mu}}]\right), \tag{2.139}$$

with

$$
\begin{aligned}
S[\psi,\bar{\psi};\lambda,A_{\bar{\mu}}] &= \int_0^{\beta_0} d\tau \left(\int d^{d-1}\bar{x}\tilde{e}\left[-\frac{1}{2}\bar{\psi}(\tilde{e}_a{}^{\bar{\mu}}\gamma^a \overrightarrow{\tilde{D}}_{\bar{\mu}} - \overleftarrow{\tilde{D}}_{\bar{\mu}}\gamma^a\tilde{e}_a{}^{\bar{\mu}})\psi - m\bar{\psi}\psi \right]\right) \\
&\equiv \int_0^{\beta_0} d\tau \int d^{d-1}\bar{x}\tilde{e}\tilde{\mathcal{L}}(\psi,\bar{\psi},\tilde{D}_\mu\psi,\tilde{D}_\mu\bar{\psi};\tilde{e}_{\bar{\mu}}{}^a,\tilde{A}_{\bar{\mu}}),
\end{aligned} \tag{2.140}
$$

where we define the (inverse) thermal vielbein $\tilde{e}_{\bar{\mu}}{}^a$ $(\tilde{e}_a{}^{\bar{\mu}})$

$$
\tilde{e}_{\bar{0}}{}^a = e^\sigma u^a, \qquad \tilde{e}_{\bar{i}}{}^a = e_{\bar{i}}{}^a, \tag{2.141}
$$

$$
\tilde{e}_a{}^{\bar{0}} = e_a{}^{\bar{0}}\frac{e^{-\sigma}}{u^{\bar{0}}}, \qquad \tilde{e}_a{}^{\bar{i}} = e_a{}^{\bar{i}} - e_a{}^{\bar{0}}\frac{u^{\bar{i}}}{u^{\bar{0}}}. \tag{2.142}
$$

Here, the thermal vielbein satisfies relations

$$
\tilde{g}_{\bar{\mu}\bar{\nu}} = \tilde{e}_{\bar{\mu}}{}^a\tilde{e}_{\bar{\nu}}{}^b\eta_{ab}, \quad \eta^{ab} = \tilde{e}_{\bar{\mu}}{}^a\tilde{e}_{\bar{\nu}}{}^b\tilde{g}^{\bar{\mu}\bar{\nu}}, \tag{2.143}
$$

and the inverse vielbein satisfies $\delta^{\bar{\nu}}_{\bar{\mu}} = \tilde{e}_{\bar{\mu}}{}^a\tilde{e}_a{}^{\bar{\nu}}$, $\delta^b_a = \tilde{e}_a{}^{\bar{\mu}}\tilde{e}_{\bar{\mu}}{}^b$. Compared with the relations such as Eq. (2.105) which the original vielbein satisfies, it is properly considered as the vielbein associated with the emergent thermal spacetime. We also introduced the covariant derivative of the imaginary time as

$$
\tilde{D}_{\bar{\mu}} \equiv \tilde{\partial}_{\bar{\mu}} - i\tilde{A}_{\bar{\mu}} - i\frac{1}{2}\tilde{\omega}_{\bar{\mu}}{}^{ab}\Sigma_{ab}, \tag{2.144}
$$

with $\tilde{A}_{\bar{\mu}} \equiv (e^\sigma\mu, A_{\bar{i}})$ and the spin connection $\tilde{\omega}_{\bar{\mu}}{}^{ab}$ in the thermal tilde space obtained from the (inverse) thermal vielbein in Eq. (2.141) and (2.142). This shows that the Masseiu-Planck functional is again written in terms of the Euclidean action in the emergent curved spacetime in the same way as fields with the integer spin. Although it is expressed by the thermal vielbein, there exists no torsion, and the structure of the emergent thermal space is completely same as the previous case.

2.4 Symmetries of emergent thermal spacetime

In the previous section, we have shown that the Masseiu-Planck functional under local thermal equilibrium for any quantum field from spin 0 to spin 1 is written in terms of a language of the emergent curved spacetime background, whose line element again has the form of the ADM metric:

$$
ds^2 = -(\tilde{N}d\tilde{t})^2 + \gamma_{\bar{i}\bar{j}}(\tilde{N}^{\bar{i}}d\tilde{t} + d\bar{x}^{\bar{i}})(\tilde{N}^{\bar{j}}d\tilde{t} + d\bar{x}^{\bar{j}}), \tag{2.145}
$$

with $\tilde{N} \equiv Nu^{\bar{0}}e^\sigma$, $\tilde{N}_{\bar{i}} = e^\sigma u_{\bar{i}}$, and $d\tilde{t} = -id\tau$. We schematically show what we have accomplished in Fig. 2.4. This gives an extension of the imaginary-time formalism in the case of local thermal equilibrium (compare Fig. 2.1 with Fig. 2.4). While the global thermal field theory is formulated under the flat Euclidean spacetime the local thermal field theory can be formulated

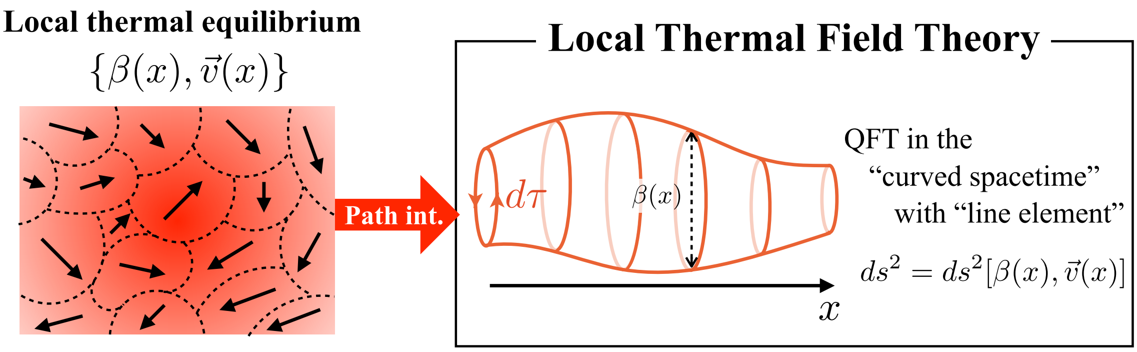


Figure 2.4: Schematic picture of the imaginary-time formalism for local thermal equilibrium.

under a curved Euclidean spacetime background. The metric $\tilde{g}_{\bar{\mu}\bar{\nu}}$, or the vielbein $\tilde{e}_{\bar{\mu}}{}^{a}$, which controls the structure of the emergent thermal spacetime, is determined by the thermodynamic parameters such as the local temperature $e^{\sigma} \equiv \beta(x)/\beta_0$, and the fluid-four velocity $u^{\mu}(x)$ as in Eq. (2.62), and thus the imaginary-time radius manifestly depends on the spatial coordinate as shown in Fig. 2.4. We, therefore, call $\tilde{g}_{\bar{\mu}\bar{\nu}}$, and $\tilde{e}_{\bar{\mu}}{}^{a}$ as the thermal metric, and the thermal vielbein. The line element, ds^2 is not real because $d\tilde{t}$ is imaginary, so that the action $S[\varphi, \lambda]$ is in general complex, which may cause the sign problem in lattice simulations. This expression of the thermal metric does not explicitly depend on the choice of the original shift vector $N^{\bar{i}}$.

We have also considered the conserved current which couples to the external $U(1)$ gauge field. It is described by the presence of a background $U(1)$ gauge connection which is slightly modified by the (local) chemical potential $\mu(x)$ as

$$\tilde{A} = \tilde{A}_{\bar{0}} d\tilde{t} + \tilde{A}_{\bar{i}} d\bar{x}^{\bar{i}}, \tag{2.146}$$

with $A_{\bar{0}} \equiv e^{\sigma}\mu$. Since the time component of the original external field does not appear in our construction, gauge invariance is only attached to the spatial component.

In this section, we demonstrate symmetries of this emergent thermal spacetime, and the background $U(1)$ gauge connection. First, we show the most prominent symmetry property related to our imaginary-time formalism, that is, the Kaluza-Klein gauge symmetry of the Masseiu-Planck functional in Sec. 2.4.1. We next see that it also has $(d-1)$-dimensional spatial diffeomorphism invariance in Sec. 2.4.2. In addition to these spacetime symmetries, we finally see the symmetric properties for the background $U(1)$ gauge connection in Sec. 2.4.3. These symmetry arguments lay out a foundation to derive hydrodynamic equations in subsequent chapters. For example, in the subsequent chapters, we will write down the possible form of the Massieu-Planck functional within the derivative expansion using above symmetric properties.

2.4.1 Kaluza-Klein gauge symmetry

First of all, we point out that the structure of the emergent thermal spacetime is invariant under the global imaginary-time translation, since the thermodynamic parameters λ^a such as the local temperature and fluid-four velocity do not depend on the imaginary time τ, and thus $\tilde{t} = -i\tau$. Furthermore, we also have a local symmetry by the spatial coordinate-dependent redefinition of the imaginary time. In order to demonstrate this symmetry, we rewrite the line element from the ADM form to the Kaluza-Klein one as

$$ds^2 = -e^{2\sigma}(d\tilde{t} + a_{\bar{i}} d\bar{x}^{\bar{i}})^2 + \gamma'_{\bar{i}\bar{j}} d\bar{x}^{\bar{i}} d\bar{x}^{\bar{j}}, \tag{2.147}$$

where we defined

$$a_{\bar{i}} \equiv -e^{-\sigma} u_{\bar{i}}, \quad \gamma'_{\bar{i}\bar{j}} \equiv \gamma_{\bar{i}\bar{j}} + u_{\bar{i}} u_{\bar{j}}. \tag{2.148}$$

Here we used $\tilde{g}_{\bar{0}\bar{0}} = -\tilde{N}^2 + \tilde{N}_{\bar{i}} \tilde{N}^{\bar{i}} = -e^{2\sigma}$. In this parametrization, the square root of determinant of the thermal metric becomes $\sqrt{-\tilde{g}} = \tilde{N}\sqrt{\gamma} = e^{\sigma}\sqrt{\gamma'}$. This parametrization of the Massieu-Planck functional was discussed in Ref. [41]. Following Refs. [41, 42], we can easily see that this metric is invariant under the local transformation (the Kaluza-Klein gauge transformation),

$$\begin{cases} \tilde{t} \to \tilde{t} + \chi(\bar{\boldsymbol{x}}), \\ \bar{\boldsymbol{x}} \to \bar{\boldsymbol{x}}, \\ a_{\bar{i}}(\bar{\boldsymbol{x}}) \to a_{\bar{i}}(\bar{\boldsymbol{x}}) - \partial_{\bar{i}} \chi(\bar{\boldsymbol{x}}), \end{cases} \tag{2.149}$$

where $\chi(\bar{\boldsymbol{x}})$ is an arbitrary function of the spatial coordinates. We note that the original induced metric $\gamma_{\bar{i}\bar{j}}$ nonlinearly transforms under this transformation since $\gamma'_{\bar{i}\bar{j}}$ does not change, so that γ is not Kaluza-Klein gauge invariant.

This symmetry enables us to restrict possible terms that appear in the construction of the Massieu-Planck functional [41]. In fact, while this symmetry does not restrict a dependence on the dilaton sector, that is the local temperature $e^{\sigma} = \beta(x)/\beta_0$, it strongly does on the thermal Kaluza-Klein gauge field $a_{\bar{i}}$. For example, $a_{\bar{i}}$ appears in the Massieu-Planck functional only through the gauge invariant combination such as the field strength $f_{\bar{i}\bar{j}}$ defined by

$$f_{\bar{i}\bar{j}} \equiv \partial_{\bar{i}} a_{\bar{j}} - \partial_{\bar{j}} a_{\bar{i}}. \tag{2.150}$$

As is shown in Sec. 2.4.3, it is also controlled by the Kaluza-Klein symmetry how the Masseiu-Planck functional depends on the external gauge field $A_{\bar{i}}$.

2.4.2 Spatial diffeomorphism symmetry

As is developed in Sec. 2.2.1, utilizing the ADM decomposition, we introduced the spatial-coordinate systems $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(x)$ on a spacelike hypersurface $\Sigma_{\tilde{t}}$. The spatial coordinate systems are described by the original induced metric $\gamma_{\bar{i}\bar{j}}$, or equivalently the modified one $\gamma'_{\bar{i}\bar{j}}$.

If we recall the simple fact that Physics does not depend on our choice of the spatial-coordinate systems $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(x)$, we can easily see that the Masseiu-Planck functional Ψ is invariant under the $(d-1)$-dimensional spatial diffeomorphism

$$\bar{\boldsymbol{x}} \to \bar{\boldsymbol{x}}'(\bar{\boldsymbol{x}}). \tag{2.151}$$

This spatial diffeomorphism invariance also restricts possible terms that could appear in the construction of the Massieu-Planck functional. For example, γ' appears only in combination with $d^{d-1}\bar{x}$, i.e., $d^{d-1}\bar{x}\sqrt{\gamma'} = d\Sigma_{\bar{t}} N e^{-\sigma}$. Note that we use $\sqrt{\gamma'}$ instead of $\sqrt{\gamma}$. This is because the modified γ' is Kaluza-Klein gauge invariant while the original one γ is not invariant.

2.4.3 Gauge connection and gauge symmetry

In the presence of the conserved $U(1)$ current coupled to the external field $A_{\bar{i}}$, we have also the background $U(1)$ gauge connection (2.146) at the same time as the emergent thermal space (2.145), or (2.147). As is already mentioned, we do not have the time-component of the original external field $A_{\bar{0}}$, and the Masseiu-Planck functional is invariant under

$$A_{\bar{i}}(\bar{\boldsymbol{x}}) \to A_{\bar{i}}(\bar{\boldsymbol{x}}) + \partial_{\bar{i}}\alpha(\bar{\boldsymbol{x}}). \tag{2.152}$$

Note that we do not have invariance under the transformation $A_{\bar{0}}(\bar{\boldsymbol{x}}) \to A_{\bar{0}}(\bar{\boldsymbol{x}}) + \partial_{\bar{t}}\alpha(\bar{\boldsymbol{x}})$.

Since $A_{\bar{i}}$ is not Kaluza-Klein gauge invariant from the same reason that the original induced metric $\gamma_{\bar{i}\bar{j}}$ is not, it is convenient to rewrite the gauge connection (2.146) in a similar way with Eq. (2.147) as follows:

$$\tilde{A} = \tilde{\mathcal{A}}_{\bar{0}}(d\tilde{t} + a_{\bar{i}}d\bar{x}^{\bar{i}}) + \tilde{\mathcal{A}}_{\bar{i}}d\bar{x}^{\bar{i}}, \tag{2.153}$$

where the modified gauge field $\tilde{A}_{\bar{\mu}}$ is defined as

$$\tilde{\mathcal{A}}_{\bar{0}} \equiv \tilde{A}_{\bar{0}} = e^{\sigma}\mu, \quad \tilde{\mathcal{A}}_{\bar{i}} \equiv \tilde{A}_{\bar{i}} - \tilde{A}_{\bar{0}}a_{\bar{i}} = A_{\bar{i}} - e^{\sigma}\mu a_{\bar{i}}. \tag{2.154}$$

From Eq. (2.153), it becomes clear that this modified gauge field $\tilde{A}_{\bar{\mu}}$ remains invariant under the Kaluza-Klein gauge transformation (2.149), since the combination $d\tilde{t} + a_{\bar{i}}d\bar{x}^{\bar{i}}$ is unchanged. Moreover, this modified background gauge field behaves in the same manner as the original one under the gauge transformation in Eq. (2.152), We, therefore, rephrase that the Masseiu-Planck functional is invariant under

$$\tilde{\mathcal{A}}_{\bar{i}}(\bar{\boldsymbol{x}}) \to \tilde{\mathcal{A}}_{\bar{i}}(\bar{\boldsymbol{x}}) + \partial_{\bar{i}}\alpha(\bar{\boldsymbol{x}}). \tag{2.155}$$

From this useful property, we use the modified gauge field $\tilde{A}_{\bar{\mu}}$ instead of the original one.

2.5 Brief summary

The main results of this chapter can be summarized as follows:

- We have introduced the local Gibbs distribution in Eqs. (2.35)-(2.36), which describes locally thermalized systems, and shown the variation formula (2.55) (Sec. 2.2).

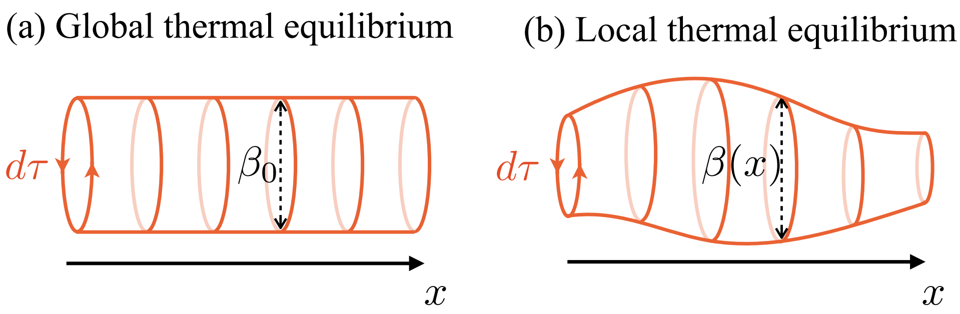


Figure 2.5: Comparison between the global thermal equilibrium (a) and local thermal equilibrium states (b). The figure is taken from [1].

- We have derived the path-integral formula of the Masseiu-Planck functional, and shown that it is written in terms of the emergent curved spacetime background (Sec. 2.3).
- We have elucidated the symmetry properties of the emergent curved spacetime: Kaluza-Klein gauge symmetry, spatial diffeomorphism symmetry, and gauge symmetry (Sec. 2.4).

The imaginary-time formalism for systems under local thermal equilibrium, in which the density operator is given by the local Gibbs distribution, is presented on the basis of the path-integral formulation. Through the detailed analysis on representative examples of quantum fields from the scalar fields to spinor fields, we have reached the conclusion that the Masseiu-Planck functional is written in terms of the path integral of the Euclidean action in the emergent curved spacetime. In Fig. 2.5, we show a schematic picture of a locally thermalized state by comparing it with that of the globally thermalized one. While the global thermal field theory is formulated under the flat Euclidean spacetime as shown in Fig. 2.5(a), the local thermal field theory can be formulated under a curved spacetime background as shown in Fig. 2.5(b).

This emergent curved spacetime has one imaginary-time direction, and $d-1$ spatial directions, and possesses the intrinsic symmetries associated with the local Gibbs distribution. They are the Kaluza-Klein gauge symmetry, the spatial diffeomorphism symmetry, and the gauge symmetry for the background gauge field. Since we also show that the Masseiu-Planck functional is regarded as the generating functional for the average values of the conserved current operator over the local Gibbs distribution, it gives a solid basis that the Masseiu-Planck functional to enjoy these symmetries in order to derive the hydrodynamic equations. This symmetry argument will be aggressively used in the subsequent chapters.

Chapter 3

Relativistic hydrodynamics

In this chapter, we derive relativistic hydrodynamics from quantum field theories by assuming that the density operator is given by a local Gibbs distribution at initial time. In order to derive hydrodynamic equations, we first rearrange the density operator into a more useful form to perform derivative expansions. This procedure is similar with the way to manage perturbative expansions in weakly interacting quantum theories based on the interaction picture. It enables us to decompose the energy-momentum tensor and conserved charge current into nondissipative and dissipative parts, and to analyze their time evolution in detail. As a result, it gives a solid basis to construct the constitutive relations order-by-order. In addition to the constitutive relations, it also brings Green-Kubo formulas for the transport coefficients such as the shear viscosity, bulk viscosity, and charge conductivity, in which they are expressed as correlation functions of conserved current operators [74, 75, 76]. We also show how useful is the path-integral formula and emergent symmetries for the Masseiu-Planck functional derived in the previous chapter to derive the nondissipative part of the constitutive relations.

This chapter is organized as follows: In Sec. 3.1, through the simple example of a classical Hamiltonian system, we first review a way to derive the second law of thermodynamics on the basis of the recent development of nonequilibrium statistical mechanics. In Sec. 3.2, considering the time evolution of hydrodynamic variables, we develop a general basis to derive hydrodynamic equations order-by-order. We also verify the second law of thermodynamics there. In Sec. 3.3, we derive leading-order constitutive relations of relativistic hydrodynamics, which reproduce a perfect fluid. In Sec. 3.4, we derive first-order dissipative correction in relativistic hydrodynamics, which gives relativistic versions of the Navier-Stokes equation. We also discuss the so-called frame ambiguity and its choice from the point of view of our setup. Sec. 3.5 is devoted to a short summary of this chapter.

The materials presented from Sec. 3.2 to Sec. 3.5 are based on our original work in collaboration with Yoshimasa Hidaka (RIKEN), Tomoya Hayata (RIKEN), and Toshifumi Noumi (Hong Kong University of Science and Technology) [1].

3.1 Basic nonequilibrium statistical mechanics

Second law of thermodynamics for isolated system

Here we illustrate how the second law of thermodynamics for a classical particle system is implemented in the modern nonequilibrium statistical mechanics [137]. Let us consider a classical N particle system trapped in the thermally isolated container with a piston which is mechanically manageable by an outside force (see Fig. 3.1). The position of the piston is denoted by $\nu(t)$. The phase space variable is $\Gamma = (\boldsymbol{r}_1, \cdots, \boldsymbol{r}_N, \boldsymbol{p}_1, \cdots, \boldsymbol{p}_N) \equiv (\{\boldsymbol{r}_i\}, \{\boldsymbol{p}_i\})$, where $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$ denote the position and momentum of $i-$th particle, respectively. The time evolution of this system is governed by Hamilton's equation of motion

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\partial H}{\partial \boldsymbol{p}_i}, \quad \frac{d\boldsymbol{p}_i}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}_i}, \quad (i = 1, \cdots, N) \tag{3.1}$$

where Hamiltonian is given by

$$H(\Gamma;\nu) = \sum_i \frac{p_i^2}{2m} + V(\{\boldsymbol{r}_i\};\nu), \quad \text{with} \quad V(\{\boldsymbol{r}_i\};\nu) = \sum_{i<j} V(\boldsymbol{r}_i - \boldsymbol{r}_j) + \sum_i V(\boldsymbol{r}_i;\nu). \tag{3.2}$$

Here $V(\{\boldsymbol{r}_i\};\nu)$ denotes the potential term composed of short range interactions between particles and also between particles and the piston, whose detailed form need not be specified in our discussion.

Then, we manipulate the position of the piston $\nu(t)$ from $t = 0$ to $t = \tau$ so that we finally restore the position of the piston

$$\nu(t) : 0 \leq t \leq \tau, \quad \nu_0 \equiv \nu(0) = \nu(\tau) \equiv \nu_1. \tag{3.3}$$

Starting from an initial phase space point Γ, the system evolves as $\Gamma \to \Gamma_\tau$ under this manipulation, which is completely determined by Hamilton's equation (3.1) with time-dependent Hamiltonian (3.2). Here we define work done by the outsider as the energy difference between initial and final states,

$$W(\Gamma) = H(\Gamma_\tau;\nu_1) - H(\Gamma;\nu_0), \tag{3.4}$$

where we distinctively write the initial and final positions of the piston as ν_0 and ν_1 although they coinside with each other: $\nu_0 = \nu_1$,

Then, the second law of thermodynamics in this situation is presented as follows.

> For the system staying in an equilibrium state, the expectation values of work under any manipulation always gives a positive value
>
> $$\langle W(\Gamma)\rangle_{\text{eq}} \geq 0, \tag{3.5}$$
>
> where $\langle \cdots \rangle_{\text{eq}}$ means the average over equilibrium states.

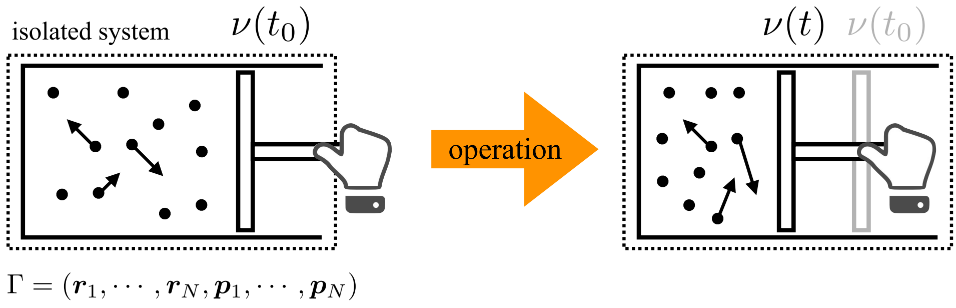


Figure 3.1: Setup for discussion of the second law of thermodynamics.

Proof of second law of thermodynamics and Jarzynski equality

Here we give a quick proof of the second law of thermodynamics based on statistical mechanics. The most crucial question here is what equilibrium states, or $\langle W(\Gamma)\rangle_{\rm eq}$, actually mean. We put an assumption that this average is regarded as the canonical Gibbs ensemble average over the initial state Γ with an inverse temperature β. In other words, we interpret $\langle W(\Gamma)\rangle_{\rm eq}$ as

$$\langle W(\Gamma)\rangle_{\rm eq} \to \langle W(\Gamma)\rangle_G = \int d\Gamma \frac{1}{Z_{\beta,\nu_0}} e^{-\beta H(\Gamma;\nu_0)} W(\Gamma), \tag{3.6}$$

where the partition function is $Z = \int d\Gamma e^{-\beta H_{\nu_0}(\Gamma)}$. Then, what we would like to show becomes the next inequality

$$\langle W(\Gamma)\rangle_G \geq 0. \tag{3.7}$$

(Proof)

In order to prove the inequality (3.7), we first evaluate the expectation values of $e^{-\beta W(\Gamma)}$.

$$\begin{aligned}
\langle e^{-\beta W(\Gamma)}\rangle_G &= \int d\Gamma \frac{1}{Z_{\beta,\nu_0}} e^{-\beta H(\Gamma;\nu_0)} e^{-\beta W(\Gamma)} \\
&= \frac{1}{Z_{\beta,\nu_0}} \int d\Gamma_\tau \left|\frac{d\Gamma}{d\Gamma_\tau}\right| e^{-\beta H(\Gamma;\nu_0)} e^{-\beta(H(\Gamma_\tau;\nu_1)-H(\Gamma;\nu_0))} \\
&= \frac{1}{Z_{\beta,\nu_0}} \int d\Gamma_\tau e^{-\beta H(\Gamma_\tau;\nu_1)} = \frac{Z_{\beta,\nu_1}}{Z_{\beta,\nu_0}},
\end{aligned} \tag{3.8}$$

where we used the definition of work and changed the integration variable to obtain the second line. To proceed to the third line, we used the Liouville's theorem, $|d\Gamma/d\Gamma_\tau| = 1$, which follows from the fact that our system is governed by Hamiltonian dynamics. Since we restore the position of the piston, taking $\nu_0 = \nu_1$, we obtain $\langle e^{-\beta W}\rangle_{\rm eq} = 1$ from the equality (3.8). Recalling that the inequality $e^{-x} \geq -x + 1$ holds for any real variable x, and putting $x = -\beta W$ we eventually obtain

$$1 = \langle e^{-\beta W(\Gamma)}\rangle_{\rm eq} \geq -\beta \langle W(\Gamma)\rangle_{\rm eq} + 1. \tag{3.9}$$

This leads to (3.7). $\square$

Noting that the partition function is given by $Z_{\beta,\nu} = e^{-\beta F(\beta,\nu)}$, where $F(\beta,\nu)$ denotes the Helmholtz free energy, we can rewrite the equality (3.8) as

$$\langle e^{-\beta W(\Gamma)} \rangle_{\rm eq} = e^{-\beta(F(\beta,\nu_1)-F(\beta,\nu_0))}. \tag{3.10}$$

This is called the Jarzynski equality [138], which relates the expectation value of any nonequilibrium work[1] to the difference of the equilibrium thermodynamic free energies. This kind of equalities and the fluctuation theorems [139, 140, 141, 142, 143, 144, 145, 146, 147] are called the nonequilibrium identities, which gives the generalization of the second law of thermodynamics.

Before closing this subsection, we make a short comment on our proof. The most important assumption in our proof is that we introduce the proper statistical ensemble only for the initial state, and the final state does not belong to the canonical one. If we permit that the final state also belongs to the canonical one, considering the inverse manipulation, we obtain the result that a similar identity holds for inverse work. This results in a failure to prove the second law of thermodynamics. Therefore, one way to implement the second law of thermodynamics for isolated systems is first to find the proper statistical ensemble, and then, to introduce it only in the initial state. Throughout this thesis, we adopt this strategy to derive the hydrodynamic equations from underlying microscopic theories.

3.2 Basis for derivative expansion

In this section, we give a way to describe the time evolution of conserved current operators, and in Sec. 3.2.1, we develop a way to describe time evolution of the expectation values of local operators. In Sec. 3.2.2, we set up a self-consistent equation which allows us to give the constitutive relations, and to evaluate the average values of the conserved current operators such as the energy-momentum tensor and conserved charge current.

3.2.1 Time evolution

Decomposition of density operator

In the previous chapter, we considered the local thermodynamics on the hypersurface. Here, we discuss the time evolution of the expectation values of local operators. In quantum field theory, the expectation value of any local operator is given by

$$\langle \hat{\mathcal{O}}(x) \rangle = \mathrm{Tr}\, \hat{\rho}_0 \hat{\mathcal{O}}(x), \tag{3.11}$$

where $\hat{\rho}_0$ is the density operator at initial time. Since we employ the Heisenberg picture, the average is always taken over the initial density operator $\hat{\rho}_0$.

[1] Note that we do not put any constraint on work like the quasi-static property, the thermodynamic second law (3.7), and Jarzynski equality (3.10) hold for any rapid manipulation of the piston.

We would like to describe the time evolution of hydrodynamic variables $c_a(x)$ in particular. If the constitutive relation is obtained, i.e., if $\langle \hat{\mathcal{J}}_a^\mu \rangle$ is expressed as a functional of c_a or conjugate variables λ^a, its time-evolution equation (hydrodynamic equation) is given by the continuity equation $\nabla_\mu \langle \hat{\mathcal{J}}_a^\mu \rangle = \delta_a^\nu F_{\nu\lambda} \langle \hat{J}^\lambda \rangle$. Therefore, our problem is how we can evaluate the average values of the conserved current operators in terms of λ^a. To obtain the constitutive relation

$$\langle \hat{\mathcal{J}}_a^\mu(x) \rangle = \mathcal{J}_a^\mu[\lambda^a(x)], \tag{3.12}$$

it is useful to decompose $\langle \hat{\mathcal{J}}_a^\mu \rangle$ into nondissipative and dissipative parts, $\langle \hat{\mathcal{J}}_a^\mu \rangle = \langle \hat{\mathcal{J}}_a^\mu \rangle_{\bar{t}}^{\mathrm{LG}} + \langle \delta \hat{\mathcal{J}}_a^\mu \rangle$. The nondissipative part $\langle \hat{\mathcal{J}}_a^\mu \rangle_{\bar{t}}^{\mathrm{LG}}$ is obviously a functional of $\lambda^a(x)$ and does not contain the information of the past state. On the other hand, we need the information of the past state to evaluate $\langle \delta \hat{\mathcal{J}}_a^\mu \rangle$. The purpose of this section is to derive the self-consistent equation to determine $\langle \delta \hat{\mathcal{J}}_a^\mu \rangle$.

At a very early stage of time evolution, the system will be far from equilibrium in a state that cannot be characterized by only thermodynamic or hydrodynamic variables. In this stage, microscopic degrees of freedom play an important role to determine the time evolution of the system. In contrast, at later times, we expect the system to be characterized by the thermodynamic variables whose time evolution is governed by the hydrodynamic equations. In this thesis, we assume that at the time $\bar{t}_0$, the density operator is given by a local Gibbs one, $\hat{\rho}_0 \equiv \hat{\rho}_{\mathrm{LG}}[\bar{t}_0; \lambda]$, although, in general, this is not exact but only approximate. As we will see below, once we assume this initial condition, the time-evolution equation can be rewritten as a compact form.

In order to evaluate the expectation value of $\hat{\mathcal{J}}_a^\mu(x)$ at the point $x^\mu \in \Sigma_{\bar{t}}$ for $\bar{t} > \bar{t}_0$, we rearrange the density operator into the new local Gibbs distribution with a new set of thermodynamic parameters $\lambda^a(x)$ on $\Sigma_{\bar{t}}$ and the other:

$$\hat{\rho}(\bar{t}_0) = \exp\bigl(-\hat{S}[\bar{t}_0; \lambda]\bigr) = \exp\bigl(-\hat{S}[\bar{t}; \lambda] + \hat{\Sigma}[\bar{t}, \bar{t}_0; \lambda]\bigr), \tag{3.13}$$

where we defined $\hat{\Sigma}[\bar{t}, \bar{t}_0; \lambda] \equiv \hat{S}[\bar{t}; \lambda] - \hat{S}[\bar{t}_0; \lambda]$. We can express $\hat{\Sigma}[\bar{t}, \bar{t}_0; \lambda]$ as the divergence of the entropy current operator as

$$\hat{\Sigma}[\bar{t}, \bar{t}_0; \lambda] = \int_{\bar{t}_0}^{\bar{t}} d\bar{s} \partial_{\bar{s}} \int d\Sigma_{\bar{s}\mu} \hat{s}^\mu = \int_{\bar{t}_0}^{\bar{t}} d\bar{s} \int d\Sigma_{\bar{s}} N \nabla_\mu \hat{s}^\mu. \tag{3.14}$$

In the last line, we used the Eq. (2.47). Let us evaluate the explicit form of $\nabla_\mu \hat{s}^\mu$. From the definition of the entropy current in Eq. (2.44), the divergence of the entropy current leads to

$$\nabla_\mu \hat{s}^\mu = -(\nabla_\mu \Lambda^a) \hat{\mathcal{J}}_a^\mu + \nabla_\mu \psi^\mu, \tag{3.15}$$

where we used the continuity equations $\nabla_\mu \hat{\mathcal{J}}_a^\mu = \delta_\nu^a F_{a\lambda} \hat{J}^\lambda$, and the definition of $\nabla_\mu \Lambda^a$ is given in Eq. (2.48). We can read the divergence of ψ^μ from Eq. (2.46) as

$$\nabla_\mu \psi^\mu = (\nabla_\mu \Lambda^a) \langle \hat{\mathcal{J}}_a^\mu \rangle_{\bar{t}}^{\mathrm{LG}}. \tag{3.16}$$

Then, the divergence of the entropy current operator reads

$$\nabla_\mu \hat{s}^\mu = -(\nabla_\mu \Lambda^a)\delta\hat{\mathcal{J}}^\mu_a = -(\nabla_\mu \beta^\nu)\delta\hat{T}^\mu_{\ \nu} - (\nabla_\mu \nu + F_{\nu\mu}\beta^\nu)\delta\hat{J}^\mu, \tag{3.17}$$

where we defined $\delta\hat{\mathcal{O}} \equiv \hat{\mathcal{O}} - \langle\hat{\mathcal{O}}\rangle^{\rm LG}_{\bar{t}}$. The entropy production rate $\langle\nabla_\mu \hat{s}^\mu\rangle$ is in general nonzero. When we decompose the expectation value of the current as $\langle\hat{\mathcal{J}}^\mu_a\rangle = \langle\hat{\mathcal{J}}^\mu_a\rangle^{\rm LG}_{\bar{t}} + \langle\delta\hat{\mathcal{J}}^\mu_a\rangle$, $\langle\hat{\mathcal{J}}^\mu_a\rangle^{\rm LG}_{\bar{t}}$ can be identified as the nondissipative part because it does not contribute to the entropy production rate, while $\langle\delta\hat{\mathcal{J}}^\mu_a\rangle$ can be identified as the dissipative part.

We will treat $\hat{\Sigma}[\bar{t}, \bar{t}_0; \lambda]$ as a perturbation term in the derivative expansion because $\nabla_\mu \hat{s}^\mu$ is proportional to the derivatives of the parameters, $\nabla_\mu \Lambda^a$. In order to expand $\hat{\rho}(\bar{t}_0)$ with respect to $\hat{\Sigma}[\bar{t}, \bar{t}_0; \lambda]$, we decompose the density operator as

$$\hat{\rho}(\bar{t}_0) = \hat{\rho}_{\rm LG}(\bar{t})\hat{U}(\bar{t}, \bar{t}_0), \tag{3.18}$$

where $\hat{U}(\bar{t}, \bar{t}_0)$ is defined as

$$\hat{U}(\bar{t}, \bar{t}_0) \equiv T_\tau \exp\left(\int_0^1 d\tau \hat{\Sigma}_\tau[\bar{t}, \bar{t}_0; \lambda]\right), \quad \text{with} \quad \hat{\Sigma}_\tau[\bar{t}, \bar{t}_0; \lambda] \equiv e^{\tau\hat{K}[\bar{t};\lambda]}\hat{\Sigma}[\bar{t}, \bar{t}_0; \lambda]e^{-\tau\hat{K}[\bar{t};\lambda]}, \tag{3.19}$$

where T_τ denotes τ ordering. The expectation value of an operator $\hat{\mathcal{O}}(x)$ on a new hypersurface $\Sigma_{\bar{t}}$ is given by

$$\langle\hat{\mathcal{O}}(x)\rangle = \langle\hat{U}\hat{\mathcal{O}}(x)\rangle^{\rm LG}_{\bar{t}}, \tag{3.20}$$

where $\langle\hat{\mathcal{O}}(x)\rangle^{\rm LG}_{\bar{t}} \equiv \operatorname{Tr}\hat{\rho}_{\rm LG}[\bar{t}; \lambda]\hat{\mathcal{O}}(x)$. As we mentioned already, this is analogous to the perturbation theory in the interaction picture in quantum field theories.

Nonequilibrium identity and the second law of thermodynamics

If one takes $\hat{\mathcal{O}} = \hat{U}^{-1}$ in Eq. (3.20), it brings about an identity $\langle\hat{U}^{-1}\rangle = 1$ corresponding to the so-called integral fluctuation theorem in nonequilibrium statistical mechanics [47]. This kind of identities and the Jarzynski equality, provide a generalization of the second law of thermodynamics, and are called the nonequilibrium identities. If we apply the Klein's inequality (see, e.g., Ref. [148]),

$$\operatorname{Tr}\hat{\rho}\ln\hat{\rho} - \operatorname{Tr}\hat{\rho}\ln\hat{\rho}' \geq 0, \tag{3.21}$$

and choosing $\hat{\rho} = \hat{\rho}(\bar{t}_0) = \hat{\rho}_{\rm LG}[\bar{t}_0; \lambda]$ and $\hat{\rho}' = \hat{\rho}_{\rm LG}[\bar{t}; \lambda]$, we find an inequality similar to the second law of thermodynamics:

$$\langle\hat{S}[\bar{t}; \lambda]\rangle - \langle\hat{S}[\bar{t}_0; \lambda]\rangle \geq 0. \tag{3.22}$$

This implies that average values of the total entropy at $\bar{t}(> \bar{t}_0)$ is always larger than that at $\bar{t}_0$. We note that this is a general result without the derivative expansion, and satisfies for arbitrary values of the parameters λ^a, but, it does not ensure the positivity of the entropy production at each time. In order to check the positivity of the entropy-current divergence at each time, we need to study $\langle\nabla_\mu \hat{s}^\mu\rangle$ using the order-by-order constitutive relations.

Entropy production formula

The entropy production operator $\hat{\Sigma}[\bar{t},\bar{t}_9;\lambda]$ is formally expressed by the use of the divergence of the entropy current in Eq. (3.14), but parameters $\lambda^a(x)$ at the point $x^\mu\in\Sigma_{\bar{t}}$ for $\bar{t}>\bar{t}_0$ is not yet specified. In fact, since Eq. (3.20) is an identity, it holds for any parameters λ^a. We, therefore, need a reasonable condition to fix the thermodynamic parameters $\lambda^a(x)$ for $\bar{t}>\bar{t}_0$. Here we impose $\langle\hat{c}_a(x)\rangle=\langle\hat{c}_a(x)\rangle^{\rm LG}_{\bar{t}}$ [56], which are explicitly given as follows:

$$n_\mu(x)\langle\hat{T}^\mu{}_\nu(x)\rangle=n_\mu(x)\langle\hat{T}^\mu{}_\nu(x)\rangle^{\rm LG}_{\bar{t}}, \tag{3.23}$$
$$n_\mu(x)\langle\hat{J}^\mu(x)\rangle=n_\mu(x)\langle\hat{J}^\mu(x)\rangle^{\rm LG}_{\bar{t}}. \tag{3.24}$$

This condition enables us to determine parameters λ^a with the help of the local thermodynamics on the new hypersurface by taking the variation of the new entropy functional through Eq. (2.43). Equations (3.23) and (3.24) also imply that the dissipative parts $\langle\delta\hat{\mathcal{J}}^\mu_a\rangle$ are orthogonal to $n_\mu(x)$, i.e., $n_\mu\langle\delta\hat{\mathcal{J}}^\mu_a\rangle=-\langle\delta\hat{c}_a\rangle=0$.

In order to consider the time evolution of the expectation values of the current operators, let us start with the spatial projection operator introduced in Sec. 2.2.1,

$$P^\mu_\nu\equiv\delta^\mu_\nu+v^\mu n_\nu\quad\text{with}\quad v^\mu n_\mu=-1\,,\quad P^\mu_\nu v^\nu=0\,,\quad P^\mu_\nu n_\mu=0\,. \tag{3.25}$$

Then, the derivative is decomposed into the time derivative and spatial derivative parts as follows

$$\nabla_\mu=(-v^\nu n_\mu+P^\nu_\mu)\nabla_\nu=-\frac{n_\mu}{N}\nabla_{\bar{t}}+\nabla_{\perp\mu}, \tag{3.26}$$

where $\nabla_{\bar{t}}=Nv^\mu\nabla_\mu$ and $\nabla_{\perp\mu}\equiv P_\mu{}^\nu\nabla_\nu$ represent the time derivative and spatial derivative, respectively. With the help of this projection operator, $\hat{\Sigma}[\bar{t},\bar{t}_0;\lambda]$ reads as

$$\hat{\Sigma}[\bar{t},\bar{t}_0;\lambda]=-\int_{\bar{t}_0}^{\bar{t}}d\bar{s}\int d\Sigma_{\bar{s}}\Big[(\nabla_{\bar{s}}\lambda^a)\delta\hat{c}_a+N(\nabla_{\perp\mu}\Lambda^a)\delta\hat{\mathcal{J}}^\mu_a\Big], \tag{3.27}$$

where we used $F_{\nu\bar{0}}\beta^\nu=F_{\nu\bar{0}}t^\nu=0$ in the hydrostatic gauge. As is clearly demonstrated in this equation, we have the time derivative of parameters $\nabla_{\bar{s}}\lambda^a$ in addition to the spatial derivatives $\nabla_{\perp\mu}\lambda^a$. We would like to eliminate this time derivative of parameters from $\hat{\Sigma}[\bar{t},\bar{t}_0;\lambda]$. As we see below, this can be performed by formally manipulating the continuity equation, $\nabla_\mu\langle\hat{\mathcal{J}}^\mu_a\rangle=\nabla_\mu\langle\hat{\mathcal{J}}^\mu_a\rangle^{\rm LG}_{\bar{t}}+\nabla_\mu\langle\delta\hat{\mathcal{J}}^\mu_a\rangle=0$.

Since $\hat{S}[\bar{t};\lambda]$ does not depend on $\bar{x}^i$, i.e., $\nabla_{\perp\mu}\hat{S}[\bar{t};\lambda]=0$, $\nabla_\mu\hat{S}[\bar{t};\lambda]=-(n_\mu/N)\partial_{\bar{t}}\hat{S}[\bar{t};\lambda]$, we can write the divergence of $\langle\hat{\mathcal{J}}^\mu_a\rangle^{\rm LG}_{\bar{t}}$ as

$$\begin{aligned}\nabla_\mu\langle\hat{\mathcal{J}}^\mu_a(x)\rangle^{\rm LG}_{\bar{t}}&=\mathrm{Tr}\Big[\frac{1}{N(x)}\big(\partial_{\bar{t}}e^{-\hat{S}[\bar{t};\lambda]}\big)\hat{c}_a(x)\Big]\\&=\frac{-1}{N(x)}\int d\Sigma'_{\bar{t}}N(x')\int_0^1 d\tau\langle e^{\tau\hat{K}[\bar{t};\lambda]}\nabla_\mu\hat{s}^\mu(x')e^{-\tau\hat{K}[\bar{t};\lambda]}\hat{c}_a(x)\rangle^{\rm LG}_{\bar{t}}\\&=\frac{1}{N(x)}\int d\Sigma'_{\bar{t}}N(x')(\nabla_\nu\Lambda^b(x'))(\delta\hat{c}_a(x),\delta\hat{\mathcal{J}}^\nu_b(x'))_{\bar{t}},\end{aligned} \tag{3.28}$$

where $(\hat{A}, \hat{B})_{\bar{t}}$ is the local Gibbs version of the Kubo-Mori-Bogoliubov inner product,

$$(\hat{A}, \hat{B})_{\bar{t}} \equiv \int_0^1 d\tau \langle e^{\hat{K}\tau} \hat{A} e^{-\hat{K}\tau} \hat{B}^\dagger \rangle_{\bar{t}}^{\rm LG}, \tag{3.29}$$

which has the following properties:

$$\text{Linearity}: \quad (a\hat{A} + b\hat{B}, \hat{C})_{\bar{t}} = a(\hat{A}, \hat{C})_{\bar{t}} + b(\hat{B}, \hat{C})_{\bar{t}}, \tag{3.30}$$

$$\text{Hermite symmetry}: \quad (\hat{A}, \hat{B})^*_{\bar{t}} = (\hat{B}, \hat{A})_{\bar{t}}, \tag{3.31}$$

$$\text{Positivity}: \quad (\hat{A}, \hat{A})_{\bar{t}} \geq 0, \quad (\hat{A}, \hat{A})_{\bar{t}} = 0 \Rightarrow \hat{A} = 0. \tag{3.32}$$

Here we used $(\langle \hat{c}_a(x) \rangle_{\bar{t}}^{\rm LG}, \delta \hat{\mathcal{J}}_b^\nu(x'))_{\bar{t}} = \langle \hat{c}_a(x) \rangle_{\bar{t}}^{\rm LG} \langle \delta \hat{\mathcal{J}}_b^\nu(x') \rangle_{\bar{t}}^{\rm LG} = 0$ to obtain the last line in Eq. (3.28). Using Eq. (3.26), we find that the continuity equation $\nabla_\mu \langle \hat{\mathcal{J}}_a^\mu \rangle = \delta_a^\nu F_{\nu\lambda} \langle \hat{J}^\lambda \rangle$ leads to

$$\begin{aligned} &\int d\Sigma'_{\bar{t}} (\delta \hat{c}_a(x), \delta \hat{c}_b(x'))_{\bar{t}} \nabla_{\bar{t}} \lambda^b(x') \\ &\quad + \int d\Sigma'_{\bar{t}} (\delta \hat{c}_a(x), \delta \hat{\mathcal{J}}_b^\nu(x'))_{\bar{t}} N(x') \nabla_{\perp\nu} \Lambda^b(x') + N(x) \nabla_\mu \langle \delta \hat{\mathcal{J}}_a^\mu(x) \rangle = 0. \end{aligned} \tag{3.33}$$

Multiplying Eq. (3.33) by the inverse of $(\delta \hat{c}_a(x), \delta \hat{c}_b(x'))_{\bar{t}}$, and integrating it with respect to the coordinates on the hypersurface, we obtain

$$\begin{aligned} \nabla_{\bar{t}} \lambda^a(x) = &- \int d\Sigma'_{\bar{t}} \int d\Sigma''_{\bar{t}} (\delta \hat{c}_a(x), \delta \hat{c}_b(x'))_{\bar{t}}^{-1} (\delta \hat{c}_b(x'), \delta \hat{\mathcal{J}}_c^\nu(x''))_{\bar{t}} N(x'') \nabla_{\perp\nu} \Lambda^c(x'') \\ &- \int d\Sigma'_{\bar{t}} (\delta \hat{c}_a(x), \delta \hat{c}_b(x'))_{\bar{t}}^{-1} N(x') \nabla_\mu \langle \delta \hat{\mathcal{J}}_b^\mu(x') \rangle. \end{aligned} \tag{3.34}$$

Let us eliminate $\nabla_{\bar{s}} \lambda^a$ in $\Sigma[\bar{t}, \bar{t}_0; \lambda]$. For this purpose, it is convenient to introduce a projection operator $\hat{\mathcal{P}}$ onto $\delta \hat{c}_a$,

$$\hat{\mathcal{P}} \hat{\mathcal{O}} = \int d\Sigma_{\bar{t}} \int d\Sigma'_{\bar{t}} \delta \hat{c}_a(x) (\delta \hat{c}_a(x), \delta \hat{c}_b(x'))_{\bar{t}}^{-1} (\delta \hat{c}_b(x'), \hat{\mathcal{O}})_{\bar{t}}. \tag{3.35}$$

This is the relativistic version of the projection operator used in Refs. [55, 149]. At global thermal equilibrium, it reduces to the Mori projection operator [150]. We have

$$(\delta \hat{c}_b(x'), \hat{\mathcal{O}})_{\bar{t}} = \frac{\delta}{\delta \lambda^b(x')} \langle \hat{\mathcal{O}} \rangle_{\bar{t}}^{\rm LG}, \tag{3.36}$$

$$(\delta \hat{c}_a(x), \delta \hat{c}_b(x'))_{\bar{t}}^{-1} = \frac{\delta \lambda^b(x')}{\delta c_a(x)}. \tag{3.37}$$

Using Eqs. (3.36) and (3.37) and the chain rule, we can rewrite Eq. (3.35) as

$$\hat{\mathcal{P}} \hat{\mathcal{O}} = \int d\Sigma_{\bar{t}} \int d\Sigma'_{\bar{t}} \delta \hat{c}_a(x) \frac{\delta \lambda^b(x')}{\delta c_a(x)} \frac{\delta}{\delta \lambda^b(x')} \langle \hat{\mathcal{O}} \rangle_{\bar{t}}^{\rm LG} = \int d\Sigma_{\bar{t}} \delta \hat{c}_a(x) \frac{\delta}{\delta c_a(x)} \langle \hat{\mathcal{O}} \rangle_{\bar{t}}^{\rm LG}. \tag{3.38}$$

Now, by using $\hat{\mathcal{P}}$, we can eliminate $\nabla_{\bar{t}}\lambda^a$ from $\hat{\Sigma}[\bar{t},\bar{t}_0;\lambda]$, and we obtain

$$\begin{aligned}\hat{\Sigma}[\bar{t},\bar{t}_0;\lambda] &= -\int_{\bar{t}_0}^{\bar{t}} d\bar{s}\int d\Sigma_{\bar{s}} N\Big[(\nabla_{\perp\mu}\Lambda^a)(1-\hat{\mathcal{P}})\delta\hat{\mathcal{J}}^\mu_a - \delta\hat{\lambda}^a\nabla_\mu\langle\delta\hat{\mathcal{J}}^\mu_a\rangle\Big] \\ &= -\int_{\bar{t}_0}^{\bar{t}} d\bar{s}\int d\Sigma_{\bar{s}} N\Big[(\nabla_{\perp\mu}\beta_\nu)\tilde{\delta}\hat{T}^{\mu\nu} + (\nabla_{\perp\mu}\nu + F_{\nu\mu}\beta^\nu)\tilde{\delta}\hat{J}^\mu - \delta\hat{\lambda}^a\nabla_\mu\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle\Big].\end{aligned} \tag{3.39}$$

Here we introduced $\tilde{\delta}\hat{\mathcal{O}} \equiv (1-\hat{\mathcal{P}})\delta\hat{\mathcal{O}}$, which enables us to remove the hydrodynamic modes from $\delta\hat{\mathcal{O}}$. In the second line, we replaced $\langle\delta\hat{\mathcal{J}}^\mu_a\rangle$ by $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle$, because the expectation value of the projected operator vanishes, $\langle\hat{\mathcal{P}}\hat{\mathcal{O}}\rangle = 0$. We also defined

$$\delta\hat{\lambda}^a(x) \equiv \int d\Sigma'_{\bar{t}}\delta\hat{c}_b(x')\frac{\delta\lambda^a(x)}{\delta c_b(x')}. \tag{3.40}$$

For later convenience, we perform the tensor decomposition for $\tilde{\delta}\hat{T}^{\mu\nu}$. Since $n_\mu\tilde{\delta}\hat{T}^{\mu\nu} = 0$ and $n_\nu\tilde{\delta}\hat{T}^{\mu\nu} = 0$, we can decompose $\tilde{\delta}\hat{T}^{\mu\nu}$ as $\tilde{\delta}\hat{T}^{\mu\nu} = h^{\mu\nu}\tilde{\delta}\hat{p} + \tilde{\delta}\hat{\pi}^{\mu\nu}$, where

$$\tilde{\delta}\hat{p} \equiv \frac{1}{d-1}h_{\rho\sigma}\tilde{\delta}\hat{T}^{\rho\sigma}, \tag{3.41}$$

$$\tilde{\delta}\hat{\pi}^{\mu\nu} \equiv P^\mu_\rho P^\nu_\sigma\tilde{\delta}\hat{T}^{\rho\sigma} - \frac{h^{\mu\nu}}{d-1}h_{\rho\sigma}\tilde{\delta}\hat{T}^{\rho\sigma}, \tag{3.42}$$

where we introduced $h^{\mu\nu} \equiv P^\mu_\rho P^\nu_\sigma g^{\rho\sigma}$ and $h_{\mu\nu}$ that satisfy $h^{\mu\rho}h_{\rho\nu} = P^\mu_\nu$.

As a result, $\hat{\Sigma}[\bar{t},\bar{t}_0;\lambda]$ reads

$$\begin{aligned}\hat{\Sigma}[\bar{t},\bar{t}_0;\lambda] = -\int_{\bar{t}_0}^{\bar{t}} d\bar{s}\int d\Sigma_{\bar{s}} N\Big[&(h^{\mu\nu}\nabla_\mu\beta_\nu)\tilde{\delta}\hat{p} + (\nabla_{\langle\mu}\beta_{\nu\rangle})\tilde{\delta}\hat{\pi}^{\mu\nu} \\ &+ (\nabla_{\perp\mu}\nu + F_{\nu\mu}\beta^\nu)\tilde{\delta}\hat{J}^\mu - \delta\hat{\lambda}^a\nabla_\mu\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle\Big],\end{aligned} \tag{3.43}$$

where we defined

$$\nabla_{\langle\mu}\beta_{\nu\rangle} \equiv \frac{P^\rho_\mu P^\sigma_\nu}{2}(\nabla_\rho\beta_\sigma + \nabla_\sigma\beta_\rho) - \frac{h_{\mu\nu}}{d-1}h^{\rho\sigma}\nabla_\rho\beta_\sigma. \tag{3.44}$$

We note that $\nabla_\mu\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle$ does not contain the explicit time derivative of the parameters, because $\nabla_\mu\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle = (-N^{-1}n_\mu\nabla_{\bar{t}} + \nabla_{\perp\mu})\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle = \big(N^{-1}(\nabla_{\bar{t}}n_\mu) + \nabla_{\perp\mu}\big)\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle$, where we used $n_\mu\nabla_{\bar{t}}\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle = -(\nabla_{\bar{t}}n_\mu)\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a\rangle$.

Since $\langle\delta\hat{\mathcal{J}}^\mu_b(x)\rangle = \langle\tilde{\delta}\hat{\mathcal{J}}^\mu_b(x)\rangle$, our goal is now to solve

$$\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_b(x)\rangle = \langle T_\tau\exp\left(\int_0^1 d\tau\hat{\Sigma}_\tau[\bar{t},\bar{t}_0;\lambda]\right)\tilde{\delta}\hat{\mathcal{J}}^\mu_b(x)\rangle^{\mathrm{LG}}_{\bar{t}}. \tag{3.45}$$

Here $\hat{\Sigma}_\tau[\bar{t},\bar{t}_0;\lambda]$ contains $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_b(x)\rangle$ as in Eq. (3.43), so that Eq. (3.45) becomes a self-consistent equation. As we discuss in the subsequent sections, $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_b(x)\rangle$ can be evaluated order-by-order in the derivative expansion with respect to the parameters.

3.2.2 Towards derivative expansion

In this subsection we lay out the way to perform the derivative expansion to derive relativistic hydrodynamic equations order-by-order. The obtained formula is used in the subsequent sections to obtain the constitutive relations up to the first order of the derivative expansion. The expectation value of the current density operator $\hat{\mathcal{J}}_a^\mu(x)$ consists of the nondissipative and dissipative parts, $\langle \hat{\mathcal{J}}_a^\mu(x)\rangle = \langle \hat{\mathcal{J}}_a^\mu(x)\rangle_{\bar{t}}^{\mathrm{LG}} + \langle \tilde{\delta}\hat{\mathcal{J}}_a^\mu(x)\rangle$.

Nondissipative part

As was shown in Sec. 2.2.3, the nondissipative part $\langle \hat{\mathcal{J}}_a^\mu(x)\rangle_{\bar{t}}^{\mathrm{LG}}$ is obtained by taking the variation of the Massieu-Planck functional $\Psi[\lambda]$ with respect to the metric $g_{\mu\nu}$, or the external gauge field $A_{\bar{\mu}}$, under the condition that the time vector $\partial_{\bar{t}}x^\mu$ is taken to coincide with the fluid-vector $e^\sigma u^\mu$: $\partial_{\bar{t}}x^\mu = e^\sigma u^\mu$. In other words, we may regard $\Psi[\lambda]$ as the generating functional for the average values of the conserved current operators over the local Gibbs distribution. Therefore, the Masseiu-Planck functional contains enough information on the nondissipative part of the constitutive relation $\langle \hat{\mathcal{J}}_a^\mu(x)\rangle^{\mathrm{LG}} = \mathcal{J}_a^\mu[\lambda^a]$.

Then, let us focus on the Massieu-Planck functional. It can be expanded as

$$\Psi[\lambda] = \sum_{n=0}^{\infty} \Psi^{(n)}[\lambda], \tag{3.46}$$

where n denotes the order of spatial derivative $O(\nabla_\perp^n)$ [2]. As was discussed in Sec. 2.4, $\Psi[\lambda]$ and therefore $\Psi^{(n)}[\lambda]$ enjoy thermal Kaluza-Klein symmetry and spatial diffeomorphism symmetry, and gauge symmetry. These symmetry arguments play an important role to construct $\Psi[\lambda]$ order-by-order. For parity symmetric theories, $\Psi^{(1)}[\lambda]$ vanishes because we cannot construct a scalar with one spatial derivative such that it is invariant under the above symmetries. On the other hand, if systems are put under the parity-violating environment such that the fermion-chirality imbalance arises, this is not true, and there exists non-zero $\Psi^{(1)}[\lambda]$ in general[3]. We consider the situation that systems has parity symmetry in this chapter. The higher-order terms are not forbidden by parity symmetry. The second or higher order hydrodynamics can contain nondissipative terms coming from them[4].

Dissipative part

The dissipative part $\langle \tilde{\delta}\hat{\mathcal{J}}_a^\mu(x)\rangle$ can be expanded as

$$\langle \tilde{\delta}\hat{\mathcal{J}}_a^\mu(x)\rangle = \sum_{m,n=0}^{\infty} \langle \tilde{\delta}\hat{\mathcal{J}}_a^\mu(x)\rangle_{(m,n)}, \tag{3.47}$$

[2] On curved space, curvatures may appear in higher-derivative terms. For example, we identify the spatial curvature as the second-order derivative, because it is given by a commutator of the spatial covariant derivatives.

[3] This is the main subject discussed in Chapter 4.

[4] In general, we can obtain such a second-order nondissipative part of the constitutive relation based on the symmetry arguments. This kind of analysis was first discussed in Ref. [41].

where the term labeled by (m,n) contains m temporal derivatives, $\nabla_{\bar{t}}$, and n spatial derivatives, $\nabla_\perp$. In order to evaluate $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle_{(n,m)}$, we expand the dissipative part $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle$ as

$$\begin{aligned}\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle &= \langle T_\tau \exp\left(\int_0^1 d\tau\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)\right)\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle^{\mathrm{LG}}_{\bar{t}} \\ &= \langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle^{\mathrm{LG}}_{\bar{t}} + \int_0^1 d\tau\langle T_\tau\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle^{\mathrm{LG}}_{\bar{t}} \\ &\quad + \frac{1}{2}\int_0^1 d\tau\int_0^1 d\tau'\langle T_\tau\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)\hat{\Sigma}_{\tau'}(\bar{t},\bar{t}_0)\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle^{\mathrm{LG}}_{\bar{t}} + \cdots .\end{aligned} \tag{3.48}$$

Here $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle^{\mathrm{LG}}_{\bar{t}}$ vanishes by definition. Since $\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)$ contains the derivative of the parameters, $\nabla_\perp\lambda^a$, $\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)$ is identified as of order $\nabla_\perp$. We note that $\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)$ does not contain the temporal derivative of the parameters, $\nabla_{\bar{t}}\lambda$. This fact implies that the derivative expansion starts from $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle_{(0,1)}$; i.e., $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle_{(l,0)}$ for $l\geq 0$ vanishes. If one considers the nth order of $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle$, one may expand Eq. (3.48) up to the nth order of $\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)$. However, we note that all correlation functions with lower orders of $\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)$ contribute to the nth order of $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle$. This is because the expansion of the average values over the local Gibbs distribution contains the derivatives of the parameters in general. For example, when we consider the derivation of the second-order hydrodynamics, in addition to the third term in the second line of Eq. (3.48), the second term also contributes to $\langle\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle_{(0,2)}$ through the derivative expansion of the correlation function $\langle T_\tau\hat{\Sigma}_\tau(\bar{t},\bar{t}_0)\tilde{\delta}\hat{\mathcal{J}}^\mu_a(x)\rangle^{\mathrm{LG}}_{\bar{t}}$. In the following, we deal with the zeroth-order and first-order hydrodynamic equations with parity symmetry.

3.3 Zeroth-order relativistic hydrodynamics: Perfect fluid

In this section, we consider the leading order of $\langle\hat{\mathcal{J}}^\mu_a\rangle$ in the derivative expansion, and show that the constitutive relations for the energy-momentum tensor and the charge current have the form of a perfect fluid.

As was discussed in Sec. 2.4, the structure of the emergent thermal space is invariant under the thermal Kaluza-Klein gauge transformation, spatial diffeomorphism, and usual gauge transformation. These symmetries strongly restrict possible terms appeared in the Masseiu-Planck functional. First, thanks to the Kaluza-Klein gauge symmetry and the gauge symmetry, $\Psi^{(0)}[\lambda]$ does not contain both of $a_{\bar{i}}$ and $A_{\bar{i}}$. Furthermore, spatial diffeomorphism invariance, together with the Kaluza-Klein gauge symmetry, restricts the γ dependence of $\Psi^{(0)}[\lambda]$ to the form proportional to $d^{d-1}\bar{\boldsymbol{x}}\sqrt{\gamma'}$. They, however, do not restrict the σ and ν dependence of $\Psi^{(0)}[\lambda]$. Then, factorizing $\Psi^{(0)}[\lambda]$, we can write down the general form of the $\Psi^{(0)}[\lambda]$ as

$$\begin{aligned}\Psi^{(0)}[\lambda] &= \int_0^{\beta_0} d\tau\int d^{d-1}\bar{x}\; e^\sigma\sqrt{\gamma'}p(\beta,\nu), \\ &= \int d^{d-1}\bar{x}\;\beta'\sqrt{\gamma}p(\beta,\nu),\end{aligned} \tag{3.49}$$

where $\beta' \equiv -n_\mu \beta^\mu$, $\beta = \beta_0 e^\sigma$, and $p(\beta, \mu)$ is the pressure of the perfect fluid as explicitly shown later. To obtain the second line, we used the relation $\beta\sqrt{\gamma'} = \beta'\sqrt{\gamma}$ and the fact that the parameters are independent of the imaginary time. Moreover, if we choose the time vector as $t^\mu = \beta^\mu$, we can equate the lapse function with β' owing to the relation $\beta' \equiv -n_\mu \beta^\mu = N$. Then, $\Psi^{(0)}[\lambda]$ is given by

$$\Psi^{(0)}[\lambda] = \int d^{d-1}\bar{x}\sqrt{-g}p(\beta, \nu). \tag{3.50}$$

Next, we consider the variation of ψ with respect to $\bar{t}$, which changes the hypersurface and n_μ. We obtain

$$d\psi = d(\beta' p) = p_\mu d\beta^\mu + n' d\nu - \beta^\mu p dn_\mu \,. \tag{3.51}$$

Again using the relation $\beta' = -n_\mu \beta^\mu$, we obtain

$$\beta' dp = (p_\mu + pn_\mu)d\beta^\mu + n' d\nu. \tag{3.52}$$

Recalling Eq. (2.55), and using this relation and the expression of $\Psi^{(0)}[\lambda]$ in Eq. (3.49), we obtain the average values of the conserved current operators as

$$\langle \hat{T}^{\mu\nu}(x)\rangle^{\mathrm{LG}}_{(0,0)} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\Psi^{(0)}[\lambda] = \left(-\frac{\beta}{\beta'}u^\rho p_\rho + p\right) u^\mu u^\nu + pg^{\mu\nu} \tag{3.53}$$

$$\langle \hat{J}^{\mu}(x)\rangle^{\mathrm{LG}}_{(0,0)} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta A_{\mu}(x)}\Psi^{(0)}[\lambda] = n'\frac{\beta}{\beta'}u^\mu, \tag{3.54}$$

where we used $\nu = \beta^\mu A_\mu$ to take the variation of $\Psi^{(0)}$ with respect to A_μ. Defining the energy density $e \equiv -\beta u^\mu p_\mu/\beta' = \langle \hat{T}^{\mu\nu}(x)\rangle^{\mathrm{LG}} u_\mu u_\nu$, and the charge density $n = n'\beta/\beta'$, we eventually obtain the zeroth-order expectation values of the energy-momentum tensor and the charge current as

$$\langle \hat{T}^{\mu\nu}(x)\rangle^{\mathrm{LG}}_{(0,0)} = (e+p)\, u^\mu u^\nu + pg^{\mu\nu} \tag{3.55}$$

$$\langle \hat{J}^{\mu}(x)\rangle^{\mathrm{LG}}_{(0,0)} = nu^\mu, \tag{3.56}$$

Equations (3.55) and (3.56) are nothing but the constitutive relations of the energy-momentum tensor and the charge current in a perfect fluid.

3.4 First-order relativistic hydrodynamics: Navier-Stokes fluid

In this section, we consider the next leading order in the derivative expansion, and derive the Navier-Stokes equation for parity-symmetric systems. In Sec. 3.4.1, we derive the first-order derivative correction to the constitutive relation with Green-Kubo formulas for the transport coefficients. We also discuss the positivity of the entropy production rate there. In Sec. 3.4.2, we discuss the frame choice, which originates from an ambiguity in the definition of the fluid four-velocity.

3.4.1 Derivation of the Navier-Stokes fluid

First-order constitutive relation with Green-Kubo formulas

First of all, it is important to note that we need not consider the nondissipative derivative corrections coming from $\langle \hat{\mathcal{J}}_a^\mu(x)\rangle_{\bar{t}}^{\mathrm{LG}}$, since $\Psi^{(1)}$ vanishes for the parity-symmetric system as is pointed out in Sec. 3.2.2. Then, Eq. (3.48) reveals that the first-order correction to the dissipative part only comes from

$$\int_0^1 d\tau \langle T_\tau \hat{\Sigma}_\tau(\bar{t},\bar{t}_0)\tilde{\delta}\hat{\mathcal{J}}_a^\mu(x)\rangle_{\bar{t}}^{\mathrm{LG}} = (\tilde{\delta}\hat{\mathcal{J}}_a^\mu(x), \hat{\Sigma}(\bar{t},\bar{t}_0))_{\bar{t}}, \tag{3.57}$$

where we used the Kubo-Mori-Bogoliubov inner product Eq. (3.29) and the Hermite symmetry of the inner product. The first-order corrections, therefore, read

$$\langle \tilde{\delta}\hat{T}^{\mu\nu}(x)\rangle_{(0,1)} \simeq h^{\mu\nu}(\tilde{\delta}\hat{p}(x), \hat{\Sigma}(\bar{t},\bar{t}_0))_{\bar{t}} + (\tilde{\delta}\hat{\pi}^{\mu\nu}(x), \hat{\Sigma}(\bar{t},\bar{t}_0))_{\bar{t}}, \tag{3.58}$$

$$\langle \tilde{\delta}\hat{J}^{\mu}(x)\rangle_{(0,1)} \simeq (\tilde{\delta}\hat{J}^{\mu}(x), \hat{\Sigma}(\bar{t},\bar{t}_0))_{\bar{t}}. \tag{3.59}$$

where $\simeq$ denotes an equality at the first order in derivatives. The right-hand side of Eqs. (3.58) and (3.59) also contain the higher-order contributions. In the first order in the derivative expansion, we can neglect $\delta\hat{\lambda}^a \nabla_\mu \langle \delta\hat{\mathcal{J}}_a^\mu\rangle$ from the entorpy production in Eq. (3.43) because $\langle \tilde{\delta}\hat{\mathcal{J}}_a^\mu\rangle = O(\nabla)$ and thus $\nabla_\mu\langle \tilde{\delta}\hat{\mathcal{J}}_a^\mu\rangle = O(\nabla^2)$. We can replace $\hat{K}$ in these inner products with $\hat{P}_\mu\beta^\mu(x)$. We remark here that the dissipative corrections are orthogonal to n_μ by construction, and thus we do not need to employ n^μ or v^ν for the tensor decomposition. Therefore, we may decompose these inner products in Eqs. (3.58) and (3.59) by only using $h^{\mu\nu}$, and two-point correlation functions with odd numbers of indices, such as $(\tilde{\delta}\hat{p}(x), \tilde{\delta}\hat{J}^\mu(x'))_{\bar{t}}$, vanish. Furthermore, correlation functions with a single $\tilde{\delta}\hat{\pi}^{\mu\nu}(x)$ also vanish since $\tilde{\delta}\hat{\pi}^{\mu\nu}(x)$ is traceless and cannot be constructed only by $h^{\mu\nu}$. In consequence, we have

$$
\begin{aligned}
(\tilde{\delta}\hat{p}(x), \hat{\Sigma}(\bar{t}, \bar{t}_0))_{\bar{t}} &= -\int_{\bar{t}_0}^{\bar{t}} d\bar{t}' \int d\Sigma_{\bar{t}'} N' (\tilde{\delta}\hat{p}(x), \tilde{\delta}\hat{p}(x'))_{\bar{t}} h^{\mu\nu}(x') \nabla_\mu \beta_\nu(x') \\
&\simeq -\frac{\zeta}{\beta(x)} h^{\mu\nu}(x) \nabla_\mu \beta_\nu(x), \qquad (3.60) \\
(\tilde{\delta}\hat{\pi}^{\mu\nu}(x), \hat{\Sigma}(\bar{t}, \bar{t}_0))_{\bar{t}} &= -\int_{\bar{t}_0}^{\bar{t}} d\bar{t}' \int d\Sigma_{\bar{t}'} N' (\tilde{\delta}\hat{\pi}^{\mu\nu}(x), \tilde{\delta}\hat{\pi}^{\rho\sigma}(x'))_{\bar{t}} \nabla_{\langle\rho}\beta_{\sigma\rangle}(x') \\
&\simeq -\frac{2\eta}{\beta(x)} h^{\mu\rho}(x) h^{\nu\sigma}(x) \nabla_{\langle\rho}\beta_{\sigma\rangle}(x), \qquad (3.61) \\
(\tilde{\delta}\hat{J}^{\mu}(x), \hat{\Sigma}(\bar{t}, \bar{t}_0))_{\bar{t}} &= -\int_{\bar{t}_0}^{\bar{t}} d\bar{t}' \int d\Sigma_{\bar{t}'} N' (\tilde{\delta}\hat{J}^{\mu}(x), \tilde{\delta}\hat{J}^{\nu}(x'))_{\bar{t}} \left(\nabla_{\perp\nu}\nu(x') + F_{\rho\nu}(x')\beta^\rho(x')\right) \\
&\simeq -\frac{\kappa}{\beta(x)} h^{\mu\nu} \left(\nabla_{\perp\nu}\nu(x) + F_{\rho\nu}(x)\beta^\rho(x)\right), \qquad (3.62)
\end{aligned}
$$

where we approximated $\nabla_\mu \lambda^a(x') \simeq \nabla_\mu \lambda^a(x)$, which is allowed in the first-order derivative expansion. Here the transport coefficients, ζ, η, and κ, are the bulk viscosity, the shear viscosity, and the diffusion constant, respectively. They are given by the Green-Kubo formulas [74, 75, 76]:

$$
\zeta = \beta(x) \int_{-\infty}^{\bar{t}} d\bar{t}' \int d\Sigma_{\bar{t}'} N' (\tilde{\delta}\hat{p}(x'), \tilde{\delta}\hat{p}(x))_{\bar{t}}, \qquad (3.63)
$$

$$
\eta = \frac{\beta(x)}{(d+1)(d-2)} \int_{-\infty}^{\bar{t}} d\bar{t}' \int d\Sigma_{\bar{t}'} N' (\tilde{\delta}\hat{\pi}^{\mu\nu}(x'), \tilde{\delta}\hat{\pi}^{\rho\sigma}(x))_{\bar{t}} h_{\mu\rho}(x) h_{\nu\sigma}(x), \qquad (3.64)
$$

$$
\kappa = \frac{\beta(x)}{d-1} \int_{-\infty}^{\bar{t}} d\bar{t}' \int d\Sigma_{\bar{t}'} N' (\tilde{\delta}\hat{J}^{\mu}(x'), \tilde{\delta}\hat{J}^{\nu}(x))_{\bar{t}} h_{\mu\nu}(x), \qquad (3.65)
$$

where we replaced $\bar{t}_0$ by $-\infty$, which can be justified in the first order in the derivative expansion. We can now obtain the first-order dissipative correction to the constitutive relation, which are given as

$$
\langle \delta \hat{T}^{\mu\nu}(x) \rangle_{(0,1)} = -\frac{\zeta}{\beta} h^{\mu\nu} h^{\rho\sigma} \nabla_\rho \beta_\sigma - 2\frac{\eta}{\beta} h^{\mu\rho} h^{\nu\sigma} \nabla_{\langle\rho}\beta_{\sigma\rangle}, \qquad (3.66)
$$

$$
\langle \delta \hat{J}^{\mu}(x) \rangle_{(0,1)} = -\frac{\kappa}{\beta} h^{\mu\nu} \left(\nabla_{\perp\nu}\nu + F_{\rho\nu}\beta^\rho\right) \qquad (3.67)
$$

Together with the nondissipative part of the constitutive relation obtained in the previous section, we obtain the complete constitutive relation up to first order as

$$\begin{aligned}\langle \hat{T}^{\mu\nu}(x)\rangle &= \langle \hat{T}^{\mu\nu}(x)\rangle^{\mathrm{LG}}_{(0,0)} + \langle \delta\hat{T}^{\mu\nu}(x)\rangle_{(0,1)} \\ &= (e+p)u^\mu u^\nu + pg^{\mu\nu} - \frac{\zeta}{\beta}h^{\mu\nu}h^{\rho\sigma}\nabla_\rho\beta_\sigma - 2\frac{\eta}{\beta}h^{\mu\rho}h^{\nu\sigma}\nabla_{\langle\rho}\beta_{\sigma\rangle}, \end{aligned} \tag{3.68}$$

$$\begin{aligned}\langle \hat{J}^{\mu}(x)\rangle &= \langle \hat{J}^{\mu}(x)\rangle^{\mathrm{LG}}_{(0,0)} + \langle \delta\hat{J}^{\mu}(x)\rangle_{(0,1)} \\ &= nu^\mu - \frac{\kappa}{\beta}h^{\mu\rho}(\nabla_\rho\nu + F_{\sigma\rho}\beta^\sigma). \end{aligned} \tag{3.69}$$

Once we calculate the transport coefficients, ζ, η, κ, and the pressure $p(\beta,\nu)$ from underlying microscopic theories, we have a set of closed equations composed of the continuity equations. These are nothing but relativistic versions of the Navier-Stokes equations. We emphasize here that we derive them without choosing a frame such as the Landau-Lifshitz frame or Eckart frame.

Positivity of entropy-current divergence

As is demonstrated in Sec. 3.2.1, we obtain the nonequilibrium identity and the corresponding inequality: $\langle \hat{S}[\bar{t};\lambda]\rangle \geq \langle \hat{S}[\bar{t}_0;\lambda]\rangle$ in Eq. (3.22). Although this is a general result without the derivative expansion, this does not mean that the entropy production rate $\langle \nabla_\mu \hat{s}^\mu\rangle$ always satisfies a positivity condition, $\langle \nabla_\mu \hat{s}^\mu\rangle \geq 0$. We, nevertheless, can show that the positivity of the entropy production rate holds in the case of the first-order derivative expansion.

Using the first-order dissipative correction to the constitutive relation in Eqs. (3.66) and (3.67), the expectation value of the entropy production rate at $\bar{t} > \bar{t}_0$ is given by

$$\begin{aligned}\langle \nabla_\mu \hat{s}^\mu\rangle &= -\Big((h^{\mu\nu}\nabla_\mu\beta_\nu)\langle\tilde{\delta}\hat{p}\rangle + (\nabla_{\langle\mu}\beta_{\nu\rangle})\langle\tilde{\delta}\hat{\pi}^{\mu\nu}\rangle + (\nabla_{\perp\mu}\nu + F_{\nu\mu}\beta^\nu)\langle\tilde{\delta}\hat{J}^\mu\rangle\Big) \\ &= \frac{\zeta}{\beta}(h^{\mu\nu}\nabla_\mu\beta_\nu)^2 + \frac{2\eta}{\beta}(\nabla_{\langle\mu}\beta_{\nu\rangle})^2 + \frac{\kappa}{\beta}(\nabla_{\perp\mu}\nu + F_{\nu\mu}\beta^\nu)^2, \end{aligned} \tag{3.70}$$

where we used a notation such as $(A_\mu)^2 \equiv A_\mu A_\nu h^{\mu\nu} = A_{\perp\mu}A_{\perp\nu}g^{\mu\nu}$. We see that the expectation value of the entropy production is give by a quadratic form of the derivative of parameters. In addition, Green-Kubo formulas for the transport coefficients from Eq. (3.63) to Eq. (3.65) tell us the positivity of them

$$\zeta > 0, \quad \eta > 0, \quad \kappa > 0. \tag{3.71}$$

Therefore, we conclude that the positivity of the entropy production rate, namely $\langle \nabla_\mu \hat{s}^\mu\rangle \geq 0$, is always satisfied for the first-order relativistic hydrodynamics. This result serves as a justification for the phenomenological derivation of the first-order hydrodynamics reviewed in Sec. 1.1.3. We note that the positivity of the entropy production rate, or the transport coefficients, is derived in our formalism, while they need to be assumed in the phenomenological derivation.

3.4.2 Choice of frame

In relativistic hydrodynamics, we face the so-called frame ambiguity, which stems from a way to define the fluid four-velocity. One useful frame is the Landau-Lifshitz frame, in which the energy flux of a fluid element vanishes at the rest frame of the fluid. Another is the Eckart frame, in which the particle flux is absent. In our approach, the choice of v^μ and n_μ corresponds to the choice of frames. In this subsection, we show that by explicitly choosing v^μ and n^μ, our constitutive relations reproduce those in the Landau-Lifshitz and Eckart frames within the derivative expansion. For simplicity, we restrict ourselves to the case without the external gauge field during this subsection.

Landau-Lifshitz frame

The fluid four-velocity in the Landau-Lifshitz frame is defined by the condition that in the local rest frame, the energy flux of a fluid element vanishes. Then, the energy and charge densities coincide with the local thermodynamic values. In other words, the Landau-Lifshitz frame is defined by [12]

$$\langle \delta\hat{T}^{\mu\nu}(x)\rangle u_{L\nu}(x) = 0, \quad \langle \delta\hat{J}^{\mu}(x)\rangle u_{L\mu}(x) = 0, \tag{3.72}$$

where the subscript L denotes the Landau-Lifshitz frame. We can easily see that Eq. (3.72) is satisfied if we choose $u_L^\mu \equiv v^\mu = n^\mu = u^\mu$. In this case, we have a familiar projection $h^{\mu\nu} = g^{\mu\nu} + u_L^\mu u_L^\nu$. The constitutive relations up to first order in the derivative expansion read

$$\langle \hat{T}^{\mu\nu}(x)\rangle = (e+p)u_L^\mu u_L^\nu + pg^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\theta h^{\mu\nu}, \tag{3.73}$$

$$\langle \hat{J}^{\mu}(x)\rangle = nu_L^\mu - \frac{\kappa}{\beta}\nabla_\perp^\mu \nu, \tag{3.74}$$

where

$$\sigma^{\mu\nu} \equiv \frac{1}{2}h^{\mu\alpha}h^{\nu\beta}(\nabla_\alpha u_{L\beta} + \nabla_\beta u_{L\alpha}) - \frac{1}{d-1}h^{\mu\nu}h^{\alpha\beta}\nabla_\alpha u_{L\beta}, \quad \theta \equiv \nabla_\mu u_L^\mu. \tag{3.75}$$

In this frame, we can explicitly write down the projected operators in Eqs. (3.63)-(3.65) as

$$\tilde{\delta}\hat{p} = \delta\hat{p} - \left(\frac{\partial p}{\partial n}\right)_e \delta\hat{n} - \left(\frac{\partial p}{\partial e}\right)_n \delta\hat{e}, \tag{3.76}$$

$$\tilde{\delta}\hat{\pi}^{\mu\nu} = \delta\hat{\pi}^{\mu\nu}, \tag{3.77}$$

$$\tilde{\delta}\hat{J}^{\mu} = \delta\hat{J}^{\mu} - \frac{n}{e+p}h^{\mu\nu}\delta\hat{p}_\nu. \tag{3.78}$$

To derive these equations, we used

$$\hat{\mathcal{P}}\delta\hat{p} = \int d\Sigma'_{\bar{t}}\delta\hat{c}_a(x')\frac{\delta}{\delta c_a(x')}\langle\hat{p}(x)\rangle^{\mathrm{LG}}_{\bar{t}} = \left(\frac{\partial p}{\partial n}\right)_e \delta\hat{n} + \left(\frac{\partial p}{\partial e}\right)_n \delta\hat{e} + O(\nabla_\perp), \tag{3.79}$$

$$\hat{\mathcal{P}}\delta\hat{J}^{\mu} = \int d\Sigma_{\bar{t}}\int d\Sigma'_{\bar{t}}\delta\hat{p}_\rho(x)\big(\delta\hat{p}_\rho(x),\delta\hat{p}_\nu(x')\big)^{-1}_{\bar{t}}\big(\delta\hat{p}_\nu(x'),\delta\hat{J}^\mu\big)_{\bar{t}} = h^{\mu\nu}\delta\hat{p}_\nu\frac{n}{e+p} + O(\nabla_\perp), \tag{3.80}$$

where $\delta\hat{e} \equiv -u^\mu_L \delta\hat{p}_\mu$, and we used the following relations [40]:

$$\int d\Sigma_{\bar{t}}(\delta\hat{p}_\rho(x), \delta\hat{p}_\nu(x'))_{\bar{t}} = \frac{1}{\beta} h_{\rho\nu}(e+p) + O(\nabla_\perp), \tag{3.81}$$

$$\int d\Sigma_{\bar{t}}(\delta\hat{p}_\nu(x), \delta\hat{J}^\mu(x'))_{\bar{t}} = \frac{n}{\beta} P^\mu_\nu + O(\nabla_\perp). \tag{3.82}$$

Eckart frame

Next, we consider the Eckart frame. The fluid four-velocity for the Eckart frame is defined by the condition that it is proportional to the particle current, i.e., $u^\mu_E(x) \equiv J^\mu(x)/\sqrt{-J^\mu(x)J_\mu(x)}$, where the subscript E denotes the Eckart frame, and $J^\mu(x) = \langle \hat{J}^\mu(x)\rangle$ [20]. It is also required that the energy density is expressed as $e = u^E_\mu \langle \hat{T}^{\mu\nu}(x)\rangle u^E_\nu(x)$. In the first order in the derivative expansion, we may choose v^μ and n^μ as

$$v^\mu = n^\mu = u^\mu_E = \frac{1}{\sqrt{-\left(u^\mu - \frac{\kappa}{\beta n}\nabla^\mu_\perp \nu\right)^2}} \left(u^\mu - \frac{\kappa}{\beta n}\nabla^\mu_\perp \nu\right) = u^\mu - \frac{\kappa}{\beta n}\nabla^\mu_\perp \nu + O(\nabla^2_\perp). \tag{3.83}$$

Using $u^\mu = u^\mu_E + (\kappa/(\beta n))\partial^\mu_\perp \nu + O(\nabla^2)$, we obtain

$$\langle \hat{T}^{\mu\nu}(x)\rangle = (e+p)u^\mu_E u^\nu_E + p g^{\mu\nu} + q^\mu u^\nu_E + u^\mu_E q^\nu - 2\eta\sigma^{\mu\nu} - \zeta\theta h^{\mu\nu}, \tag{3.84}$$

$$\langle \hat{J}^\mu(x)\rangle = n u^\mu_E, \tag{3.85}$$

where we dropped the terms of order $\nabla^2_\perp$. $\sigma^{\mu\nu}$ and θ are obtained by replacing u^μ_L in Eq. (3.75) with u^μ_E. The thermal conductivity q^μ, which is absent in the Landau-Lifshitz frame reads

$$q^\mu = \frac{e+p}{n\beta}\kappa \nabla^\mu_\perp \nu. \tag{3.86}$$

We note that the shear and bulk viscous terms are the same as those of the Landau-Lifshitz frame.

Although we do not have the charge diffusion in this frame, the expression of heat current is slightly different from the original Eckart one q^μ_E, which is given by [20]

$$q^\mu_E = -\lambda(\nabla^\mu_\perp T + T\nabla_{\bar{t}} u^\mu_E), \tag{3.87}$$

where λ denotes the thermal conductivity of the fluid. The apparent difference is coming from whether we use the time derivative of the fluid four-velocity in order to construct the constitutive relations. Although we utilize the Mori projection operator to eliminate the time derivative of the parameters from the entropy production, we can reconstruct the constitutive relations by using the time derivative terms with the help of the equation of motion. In the first order, we can use the equation of motion for the perfect fluid,

$$\nabla_{\bar{t}} u^\mu = -\frac{1}{T}\nabla^\mu_\perp T - \frac{nT}{e+p}\nabla^\mu_\perp \nu, \tag{3.88}$$

in order to eliminate $\nabla^\mu_\perp \nu$ from Eq. (3.86). Then, we derive the constitutive relations in the original Eckart frame with $\lambda = ((e+p)^2\beta/n^2)\kappa$.

Obviously, in our formalism, the constitutive relations in the Landau-Lifshitz and Eckart frames are equivalent within the first order in the derivative expansion. These are related to each other by the redefinition of the fluid four-velocity, $u^\mu_L \leftrightarrow u^\mu_E + (\kappa/(\beta n))\nabla^\mu_\perp \nu$ in Eqs. (3.73) and (3.74). More generally, if we choose a frame such that $v^\mu = u^\mu + O(\nabla)$ and $n^\mu = u^\mu + O(\nabla)$, the constitutive relations in this frame are equivalent to those in the Landau frame within the first order in the derivative expansion. Namely, if n^μ is a functional of λ^a, the constitutive relations are unique and become those in the Landau-Lifshitz frame. We note that such a uniqueness was also discussed in Ref. [151] based on the Boltzmann equation.

3.5 Brief summary

The main results of this chapter can be summarized as follows:

- Assuming the local Gibbs distribution at initial time, we have derived the self-consistent equation for the dissipative part of constitutive relations (3.45) (Sec. 3.2).
- We have derived the zeroth-order constitutive relations (3.55)-(3.56), which result in ones for a perfect fluid, with the equation of state (3.50) (Sec. 3.3).
- We have derived the first-order constitutive relations (3.68)-(3.69), which bring about the relativistic Navier-Stokes equation, with the Green-Kubo formula for transport coefficients (3.63)-(3.65) (Sec. 3.4).

Relativistic hydrodynamic equations for systems with the parity symmetry are derived based on quantum field theories. In order to derive them we put an assumption that the initial density operator has a form of the local Gibbs distribution, and lay out a way to describe the time evolution of the hydrodynamic variables with the help of the rearrangement of this density operator. This manipulation provides us the self-consistent equation for the expectation values of the conserved current operators in the case including the deviation from local thermal equilibrium.

After the elaborate preparation of the self-consistent equation to obtain the constitutive relations, we perform the derivative expansion on the top of the local Gibbs distribution newly introduced at later time. On the basis of the symmetry argument obtained in the previous chapter, we first construct the leading-order Masseiu-Planck functional for parity-even systems Together with the derivative expansion of the self-consistent equation, we derive the zeroth-order and the first-order relativistic constitutive relations, which result in one of the perfect fluid, and the Navier-Stokes fluid, respectively. In addition to the constitutive relations, we obtain the Green-Kubo formulas, in which the quantum field theoretical expression for the transport coefficients such as the shear viscosity are given.

The real-time evolution in our formulation is schematically shown in Fig. 3.2. The density operator of the system at initial time $\bar{t}_0$ is assumed to have the form of the local Gibbs distribution. Then we expand the density operator at a later time $\bar{t}$ around the new local Gibbs distribution with the thermodynamic parameters $\lambda^a(x)$ at that time. In each time, the local Gibbs distribution (the Massieu-Planck functional) can be expressed by using the imaginary-time path integral under the curved spacetime background $\Sigma_{\bar{t}} \times S^1$, whose metric is given in Eq. (2.62). After a sufficiently long time, the system reaches the global thermal equilibrium with the uniform imaginary-time radius β_0. The local Gibbs distribution enables us to treat a nonequilibrium state beyond the real-time formalism [123], in which the distribution is necessarily in the global equilibrium. However, in an early stage far from equilibrium, the density operator cannot be approximated by the local Gibbs distribution, and thus our formulation is no longer applicable.

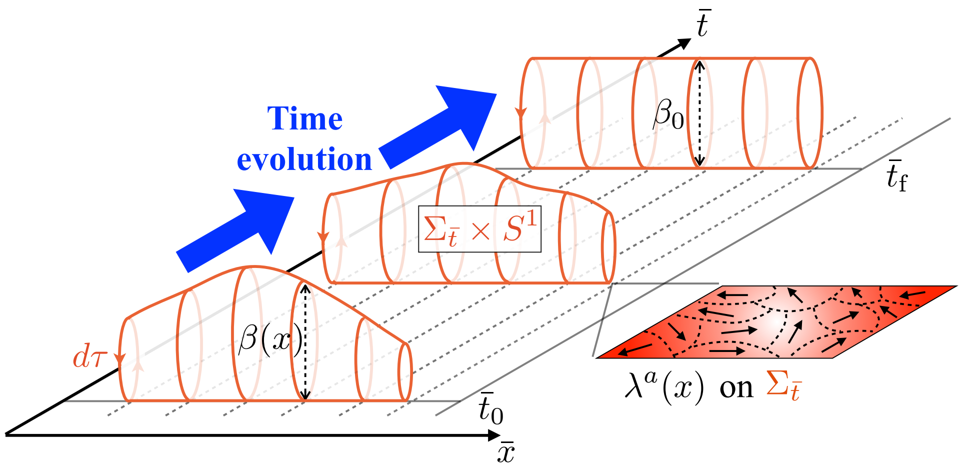


Figure 3.2: Schematic figure of the real-time evolution in our formulation toward the global thermal equilibrium. The figure is taken from [1].

Chapter 4

Anomalous hydrodynamics

In this chapter, we derive the anomaly-induced transport based on our formulation developed in Chapter 2. For this purpose, we consider a system composed of the Weyl fermion coupled to external gauge fields, which has the quantum anomaly. As is discussed in Chapter 2, information on the nondissipative part of the constitutive relation is fully contained in the Masseiu-Planck functional written in terms of the path-integral formula under the background curved spacetime with the background gauge connection. We first show the zeroth-order and the first-order dissipative parts of the constitutive relations are unchanged even if there exists the quantum anomaly. Then, we show that the quantum anomaly causes the first-order nondissipative corrections thorough the first-order anomalous corrections to the Masseiu-Planck functional. Following the discussion in Refs. [41, 86], we discuss the possible form of the first-order Masseiu-Planck functional with the help of the symmetry argument. Then, we proceed to perturbative calculation, and evaluate the first-order nondissipative correction to the Masseiu-Planck functional at one-loop level, from which we can read off the constitutive relation for the anomalous current. As a consequence, we obtain the anomaly-induced transport phenomena such as the chiral magnetic effect, the chiral separation effect, and the chiral vortical effect.

This chapter is organized as follows: In Sec. 4.1, we first demonstrate what is affected and is not affected by the quantum anomaly, and show the possible form of the Masseiu-Planck functional in the first-order derivative expansion. In Sec. 4.2, we evaluate the first-order anomalous corrections to the Masseiu-Planck functional in the perturbative way, which results in the anomaly-induced transport phenomena. Sec. 4.3 is devoted to a short summary of this chapter.

Sec. 4.2 is based on our original work in collaboration with Yoshimasa Hidaka (RIKEN) [136].

4.1 Hydrodynamics in the presence of anomaly

In this section, we generalize our discussion on the derivation of the hydrodynamic equation in the presence of the quantum anomaly. In Sec. 4.1.1, we generalize our derivation of hydrodynamic equations, and show the place where the anomaly-induced transport arises. In Sec. 4.1.2, we show a possible form of the first-order Masseiu-Planck functional based on the symmetry argument.

4.1.1 Absence of the first-order anomalous dissipative transport

Suppose that the system considered is composed of the chiral fermions and couples to external gauge fields. Such a system contains the quantum anomaly [90, 91, 92]. For example, let us consider the system composed of the right-handed Weyl fermions in the $d = 4$ spacetime dimension. Then, in addition to the energy-momentum tensor, which is not conserved by the Lorentz force, the right-handed current is no longer conserved:

$$\nabla_\mu \hat{T}^{\mu\nu} = F_{\mu\nu} \hat{J}^\mu, \tag{4.1}$$

$$\nabla_\mu \hat{J}^\mu_R = C_R \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{4.2}$$

where $F_{\mu\nu}$ denotes the field strength of the external gauge field, $\varepsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric tensor, and C_R the anomaly coefficient. Strictly speaking, the right-handed current is not the conserved one due to the quantum anomaly, and, thus, it seems meaningless to introduce the chemical potential for such a non-conserved charge. Nevertheless, there is a situation in which we can approximately introduce the chemical potential for the divergent current e.g. the QGP in heavy-ion collisions[1]. Then, we treat the energy-momentum and right-handed current as the conserved quantities, and introduce the local Gibbs distribution. In this case, $\hat{K}[\bar{t};\lambda]$ formally takes the same form as before, in which the chemical potential $\nu = \beta\mu$ for the vector charge is replaced by the one for the right-handed charge $\nu_R = \beta\mu_R$.

Using Eqs. (4.1) and (4.2) instead of Eqs. (1.2) and (1.3), the divergence of the entropy current operator reads

$$\nabla_\mu \hat{s}^\mu = -(\nabla_\nu \beta^\mu)\delta \hat{T}^\mu_{\ \nu} - (\nabla_\mu \nu_R + \beta^\nu F_{\mu\nu})\delta \hat{J}^\mu_R(x), \tag{4.3}$$

where $\delta\hat{\mathcal{O}} \equiv \hat{\mathcal{O}} - \langle \hat{\mathcal{O}} \rangle^{\mathrm{LG}}_{\bar{t}}$ is the same one encountered in the previous chapter. Although we have the nonzero divergence (4.2) due to the quantum anomaly, we have no contribution in $\nabla_\mu \hat{s}^\mu$. This is because the term $C_R \nu_R \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ coming from the anomaly cancels out in the divergence of the entropy current operator. Therefore, the anomaly does not directly contribute to the dissipative part of the currents, and the first-order dissipative part of the constitutive relations remains unchanged. This is consistent with the observations found in the entropy-production method [59] and in the generating-functional method [41, 85, 86, 87, 88].

Let us now move onto the nondissipative part of the constitutive relations. Full information on the nondissipative part of the constitutive relations is still contained in the Masseiu-Planck

[1] The QGP experience the rapid expansion governed by the relativistic hydrodynamics, whose typical time scale is given by the QCD scale ($\sim 1\,\mathrm{fm/c} \sim 200\,\mathrm{MeV} \sim 10^{-23}\,\mathrm{s}$). On the other hand, a time scale of the breakdown of conservation laws caused by the quantum anomaly associated with the $U(1)$ gauge field is much longer than it due to the small fine structure constant. The existence of this scale separation allows us to introduce the chemical potential for the anomalous current.

functional. Then, we focus on the possible corrections to the Masseiu-Planck functional. In the completely same way as given in Chapter. 3, the symmetry properties of the Masseiu-Planck functional play an important role: The Kaluza-Klein gauge symmetry and gauge symmetry for the external gauge field prohibit us from constructing the zeroth-order terms in terms of the gauge fields $A_{\bar{i}}$ and $a_{\bar{i}}$ even if there exists the quantum anomaly. Therefore, the possible anomalous corrections appear only in the next-to-leading corrections to the Masseiu-Planck functional.

4.1.2 Anomalous correction to the Masseiu-Planck functional

We have laid out the symmetry properties of the Masseiu-Planck functional in Sec. 2.4. Following Refs. [41, 86], we show a possible modification of the first-order Masseiu-Planck functional based on the symmetry argument.

As shown in Chapter 3, we see that the Kaluza-Klein gauge field $a_{\bar{i}}$ and the modified external gauge field $\tilde{\mathcal{A}}_{\bar{i}}$ do not appear in the next-to-leading-order expressions of the Masseiu-Planck functional for the parity-symmetric systems. This is a macroscopic manifestation of the parity invariance of the systems. However, the situation is changed if the system does not have parity invariance due to parity-violating variables such as the chemical potential for the right-handed fermions μ_R.

In our notation, it is written as

$$\Psi^{(1)}[\lambda, A_{\bar{\mu}}] = \int d^3x\sqrt{\gamma'}\varepsilon^{ijk}\left[C\left(\frac{\nu_R}{3}\tilde{\mathcal{A}}_i\partial_j\tilde{\mathcal{A}}_k + \frac{\nu_R e^{\sigma}\mu_R}{6}\tilde{\mathcal{A}}_i\partial_j a_k\right) + T_0 C_3\tilde{\mathcal{A}}_i\partial_j a_k\right], \tag{4.4}$$

where $T_0 \equiv 1/\beta_0$ denotes the temperature, and C the anomaly coefficient determined by the consistency with the quantum anomaly. Here C_3 is also constant, but is not determined by the quantum anomaly[2]. Though the symmetry argument restricts the possible form as above and the coefficient C, we cannot determine C_3 without another consideration [152, 105, 106, 87, 153, 88]. We evaluate them on the basis of the perturbative approach in the next section.

4.2 Derivation of the anomaly-induced transport

In this section, we provide a perturbative approach to calculate the Masseiu-Planck functional under the external fields. In Sec. 4.2.1, we set out the basis for the perturbative calculation of the Masseiu-Planck functional. In Sec. 4.2.2, we evaluate the first-order corrections to the Masseiu-Planck functional of the system with Weyl fermions. In Sec. 4.2.3, we derive the anomaly-induced transport.

[2] If the systems considered do not have the CPT invariance, another term could appear as discussed in [41]. Here, we do not consider such terms for sake of simplicity.

4.2.1 Perturbative approach to the Masseiu-Placnk functional

Let us consider the system in local thermal equilibrium. As is discussed in Chapter 2, the Massieu-Planck functional is written in terms of the path integral of the Euclidean action $S[\varphi;\tilde{g}_{\bar{\mu}\bar{\nu}},\tilde{A}_{\bar{\mu}}]$. Then, we expand $S[\varphi;\tilde{g}_{\bar{\mu}\bar{\nu}},\tilde{A}_{\bar{\mu}}]$ around the one in global thermal equilibrium described by $\tilde{g}^{(\mathrm{eq})}_{\bar{\mu}\bar{\nu}}$ and $\tilde{A}^{(\mathrm{eq})}_{\bar{\mu}}$:

$$S[\varphi;\tilde{g}_{\bar{\mu}\bar{\nu}},\tilde{A}_{\bar{\mu}}]=S_0[\varphi;\tilde{g}^{(\mathrm{eq})}_{\bar{\mu}\bar{\nu}},A^{(\mathrm{eq})}_{\bar{\mu}}]+\delta S[\varphi;\tilde{g}_{\bar{\mu}\bar{\nu}},\tilde{A}_{\bar{\mu}}]. \tag{4.5}$$

In the linear order for the external fields, we can write the coupling part as

$$\delta S[\varphi;\tilde{g}_{\bar{\mu}\bar{\nu}},\tilde{A}_{\bar{\mu}}]=\int_0^{\beta_0}d\tau\int d^{d-1}\bar{x}\sqrt{-\tilde{g}}\left(\delta\tilde{A}_{\bar{\mu}}\tilde{J}^{\bar{\mu}}(\tau,\boldsymbol{x})+\frac{1}{2}\delta\tilde{g}_{\bar{\mu}\bar{\nu}}\tilde{T}^{\bar{\mu}\bar{\nu}}(\tau,\boldsymbol{x})\right), \tag{4.6}$$

with $\delta\tilde{g}_{\bar{\mu}\bar{\nu}}\equiv\tilde{g}_{\bar{\mu}\bar{\nu}}-\tilde{g}^{(\mathrm{eq})}_{\bar{\mu}\bar{\nu}}$ and $\delta\tilde{A}_{\bar{\mu}}\equiv A_{\bar{\mu}}-A^{(\mathrm{eq})}_{\bar{\mu}}$. Here we introduced the energy-momentum tensor and the current in the thermal space as follows:

$$\tilde{T}^{\bar{\mu}\bar{\nu}}(\tau,\boldsymbol{x})\equiv\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S}{\delta\tilde{g}_{\bar{\mu}\bar{\nu}}(\tau,\boldsymbol{x})},\quad \tilde{J}^{\bar{\mu}}(\tau,\boldsymbol{x})\equiv\frac{1}{\sqrt{-\tilde{g}}}\frac{\delta S}{\delta\tilde{A}_{\bar{\mu}}(\tau,\boldsymbol{x})}. \tag{4.7}$$

When we consider the fermionic systems we define the energy-momentum tensor by taking a variation of the Euclidean action with respect to the thermal vierbein instead of the thermal metric,

$$\tilde{T}^{\bar{\mu}\bar{\nu}}(\tau,\boldsymbol{x})\equiv\frac{1}{\tilde{e}}\left(\tilde{e}_a^{\ \bar{\nu}}\frac{\delta S}{\delta\tilde{e}_{\bar{\mu}}^{\ a}}+\tilde{e}_a^{\ \bar{\mu}}\frac{\delta S}{\delta\tilde{e}_{\bar{\nu}}^{\ a}}\right), \tag{4.8}$$

Here local Lorentz invariance allows us to adopt the symmetric energy-momentum tensor. We note that these currents still depend on the thermal metric and the external gauge field in general.

Then, we consider the perturbative expansion of the Masseiu-Planck functional with respect to the external fields:

$$\Psi[\lambda]=\log Z=\Psi^{(0)}[\lambda]+\Psi^{(1)}[\lambda]+\mathcal{O}((\partial_\perp)^2), \tag{4.9}$$

where $\Psi^{(0)}[\lambda]$ denotes the leading-order Masseiu-Planck functional in Eq. (3.49). The next-to-leading-order Masseiu-Planck functional $\Psi^{(1)}[\lambda]$ is given by

$$\begin{aligned}\Psi^{(1)}[\lambda]=&\frac{1}{2}\int_0^{\beta_0}d\tau_1 d\tau_2\int d^{d-1}x_1 d^{d-1}x_2\\&\times\left(\langle\tilde{J}^{\bar{\mu}}(X_1)\tilde{J}^{\bar{\nu}}(X_2)\rangle_0\delta\tilde{A}_{\bar{\mu}}(X_1)\delta\tilde{A}_{\bar{\nu}}(X_2)+\langle\tilde{T}^{\bar{\mu}\bar{\nu}}(X_1)\tilde{J}^{\bar{\alpha}}(X_2)\rangle_0\delta\tilde{g}_{\bar{\mu}\bar{\nu}}(X_1)\delta\tilde{A}_{\bar{\alpha}}(X_2)+\cdots\right),\end{aligned} \tag{4.10}$$

with $X\equiv(\tau,\boldsymbol{x})$. Here $\langle\cdots\rangle_0$ denotes the path integral over the action $S_0[\varphi;\tilde{g}^{(\mathrm{eq})}_{\bar{\mu}\bar{\nu}},\tilde{A}^{(\mathrm{eq})}_{\bar{\mu}}]$, which only contains the connected diagrams.

4.2.2 Masseiu-Planck functional for Weyl fermion

Here we consider the system composed of the right-handed Weyl fermions ξ under the external $U(1)$ gauge field in the $d=4$ spacetime dimension, whose Lagrangian reads

$$\mathcal{L} = \frac{i}{2}\xi^\dagger \left(e_a{}^\mu \sigma^a \overrightarrow{D}_\mu - \overleftarrow{D}_\mu \sigma^a e_a{}^\mu \right) \xi, \tag{4.11}$$

where we introduced $\sigma^a = (1, \sigma^i)$ with the Pauli matrices σ^i $(i = 1, 2, 3)$. Since this Lagrangian describes the chiral fermion, we have the quantum anomaly

$$\partial_\mu \hat{J}^\mu_R = C_R \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad \text{with} \quad \hat{J}^\mu_R = \xi^\dagger \sigma^\mu \xi, \tag{4.12}$$

where C_R denotes the anomaly coefficient, which results in $C_R = 1/(32\pi^2)$ for single component Weyl fermion. As is discussed in the previous section, we introduce the chemical potential for this divergent current, which results in the Euclidean action

$$S[\xi, \xi^\dagger; \lambda, A_{\bar{\mu}}] = \int_0^{\beta_0} d\tau \int d^3\bar{x}\, \tilde{e} \left[\frac{i}{2}\xi^\dagger \left(\tilde{e}_a{}^{\bar{\mu}} \sigma^a \overrightarrow{\tilde{D}}_{\bar{\mu}} - \overleftarrow{\tilde{D}}_{\bar{\mu}} \sigma^a \tilde{e}_a{}^{\bar{\mu}} \right) \xi \right], \tag{4.13}$$

where $\tilde{e}_a{}^\mu$ is the inverse thermal vierbein defined in Eq. (2.142), and $\tilde{D}_{\bar{\mu}}$ the covariant derivative in the thermal tilde space in Eq. (2.144). By taking the variations with respect to the Euclidean action, we construct the symmetric energy-momentum tensor and the right-handed fermion current in the thermal space as

$$\tilde{T}^{\bar{\mu}\bar{\nu}}(X) = -\frac{i}{4}\xi^\dagger(X)(\sigma^{\bar{\mu}} \overrightarrow{\tilde{D}}^{\bar{\nu}} + \sigma^{\bar{\nu}} \overrightarrow{\tilde{D}}^{\bar{\mu}} - \overleftarrow{\tilde{D}}^{\bar{\nu}} \sigma^{\bar{\mu}} - \overleftarrow{\tilde{D}}^{\bar{\mu}} \sigma^{\bar{\nu}})\xi(X) + \tilde{g}^{\bar{\mu}\bar{\nu}} \mathcal{L}, \tag{4.14}$$

$$\tilde{J}^{\bar{\mu}}_R(X) = \xi^\dagger(X) \sigma^{\bar{\mu}} \xi(X), \tag{4.15}$$

Suppose that the charge density has a uniform distribution, while other external fields have smooth spatial coordinate dependence. In other words, we treat the chemical potential in a nonperturbative way by making $S_0[\varphi; \beta_0, \mu_R]$ contain the constant chemical potential, while other parts of the external fields are treated perturbatively:

$$S[\xi, \xi^\dagger; \tilde{g}_{\bar{\mu}\bar{\nu}}, \tilde{A}_{\bar{\mu}}] = S_0[\xi, \xi^\dagger, \beta_0, \mu_R] + \delta S[\varphi; \tilde{g}_{\bar{\mu}\bar{\nu}}, \tilde{A}_{\bar{i}}], \tag{4.16}$$

Furthermore, for sake of simplicity, we assume that the original spacetime is flat, and choose the cartesian coordinate system. Then, we can replace the covariant derivative and the thermal metric in the currents (4.14) and (4.15) with the partial derivative and Minkowski metric in the leading-order derivative expansion.

By performing the Fourier transformation, we rewrite the Euclidean action $S_0[\xi, \xi^\dagger, \mu_R]$ as

$$S_0[\xi, \xi^\dagger] = -\sum_P \xi^\dagger(P) \sigma^\mu \tilde{P}_\mu \xi(P) \equiv -\sum_P \xi^\dagger_a(P) \left(\mathcal{G}_0^{-1}(\tilde{P}) \right)_{ab} \xi_b(P), \tag{4.17}$$

where $a, b (= 1, 2)$ denote the spinor indices. We introduced the Fourier transformation as

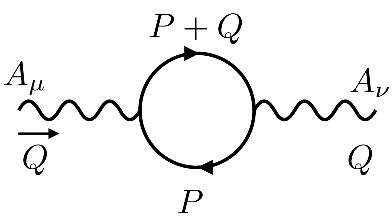
$$\xi(X) = T_0 \sum_{\omega_n} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} e^{-i\omega_n \tau + i\boldsymbol{p}\cdot\boldsymbol{x}} \xi(\omega_n, \boldsymbol{p}) \equiv \sum_P e^{iP\cdot X} \xi(P), \tag{4.18}$$

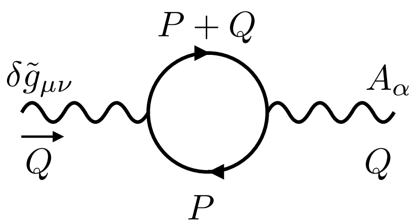
with the temperature $T_0 \equiv 1/\beta_0$. Here we defined $\tilde{P}_\mu \equiv (-i\omega_n - \mu_R, \boldsymbol{p})$ with the chemical potential μ_R. We also introduced the free propagator $\mathcal{G}_0(P)$

$$\mathcal{G}_0^{-1}(P) \equiv \sigma^\mu P_\mu, \quad \mathcal{G}_0(P) = \frac{\bar{\sigma}^\mu P_\mu}{P^2}, \tag{4.19}$$

with $\bar{\sigma}^a \equiv (-1, \sigma^i)$ and $\sigma^a \bar{\sigma}^b + \sigma^b \bar{\sigma}^a = 2\eta^{ab}$. Note that the argument of the propagator in Eq. (4.17) is not P but $\tilde{P}$, and, thus, it represents the propagator fully dressed by the chemical potential μ_R.

We, then, evaluate the first-order Masseiu-Planck functional (4.10) related to the anomaly-induced transport phenomena. The first-order Masseiu-Planck functional is divided into three sectors: the two-point function between currents, and the two-point function between the energy-momentum tensor and the current, and the two-point function between the energy-momentum tensors. Since we are now interested in the anomalous current, we focus on the first two sectors, which contain at least one external gauge field A_i. In other words, we only focus on the following diagrams:



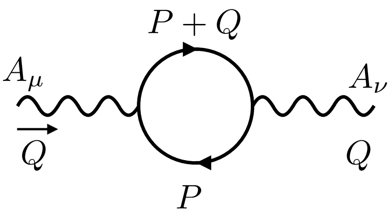
and


, (4.20)

where we will take the long-wave-length limit $Q \sim 0$.

Evaluation of the anomalous Masseiu-Planck functional

First, let us evaluate the term which contains two external gauge fields:

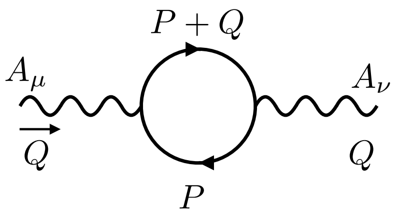


$$= -T_0 \sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \operatorname{tr} \left(\frac{(\tilde{Q}_\rho + \tilde{P}_\rho)\tilde{P}_\sigma \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\sigma \sigma^\nu}{(\tilde{Q} + \tilde{P})^2 \tilde{P}^2} \right), \tag{4.21}$$

where we used the free propagator defined in Eq. (4.19). Here "tr" denotes the trace over the spinor indices. With the help of the trace formula for the Pauli matrices

$$\operatorname{tr} \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\alpha \sigma^\beta = -2i\varepsilon^{\mu\nu\alpha\beta} + 2\eta^{\mu\nu}\eta^{\alpha\beta} - 2\eta^{\mu\alpha}\eta^{\nu\beta} + 2\eta^{\mu\beta}\eta^{\nu\alpha}, \tag{4.22}$$

we can decompose the two-point functions into the totally antisymmetric part and other parts. Since we are interested in the anomalous term which results from the totally antisymmetric part, we focus on that part:



$$= -T_0 \sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \operatorname{tr}\left(\frac{(\tilde{Q}_\rho + \tilde{P}_\rho)\tilde{P}_\sigma \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\sigma \sigma^\nu}{(\tilde{Q}+\tilde{P})^2 \tilde{P}^2} \right) \tag{4.23}$$

$$= 2i\varepsilon^{\rho\mu\sigma\nu}\tilde{Q}_\rho \delta^0_\sigma B_3^{(1,0)}(0) + (\text{symmetric terms}) + \mathcal{O}(Q^2),$$

Here we have introduced

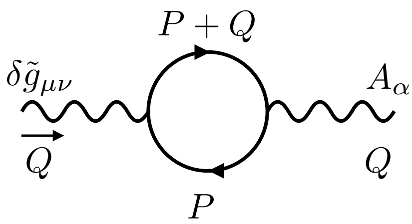
$$B_3^{(k,l)}(m) \equiv T_0 \sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{(\tilde{P}_0)^k (\boldsymbol{p})^l}{(\tilde{P}^2 + m^2)^2}. \tag{4.24}$$

As evaluated in Appendix. A.2, we have $B_3^{1,0}(0) = \mu_R/(8\pi^2)$, which leads to

$$\begin{aligned} \Psi_1^{(1)}[\lambda] &= \frac{1}{2}\sum_Q \frac{\mu_R}{4\pi^2} i\varepsilon^{\rho\mu\sigma\nu}\tilde{Q}_\rho \delta^0_\sigma A_\mu(Q)A_\nu(-Q) \\ &= \int_0^{\beta_0} d\tau \int d^3\boldsymbol{x} \frac{\mu_R}{8\pi^2}\varepsilon^{0ijk} A_i(\boldsymbol{x})\partial_j A_k(\boldsymbol{x}) \end{aligned} \tag{4.25}$$

Since the external gauge field does not depend on the imaginary time, we perform the imaginary-time integral explicitly.

Next, let us evaluate the term which contains one external gauge field and one thermal metric. Then, a similar calculus brings about the next result



$$= -\frac{1}{2}T_0 \sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} (2\tilde{P}^\nu + \tilde{Q}^\nu) \operatorname{tr}\left(\frac{(\tilde{Q}_\rho + \tilde{P}_\rho)\tilde{P}_\sigma \bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\sigma \sigma^\alpha}{(\tilde{Q}+\tilde{P})^2 \tilde{P}^2} \right).$$

$$= 2i\tilde{Q}_\rho \left(\eta^{\nu 0}\varepsilon^{\rho\mu 0\alpha} B_3^{(2,0)}(0) + \frac{1}{3}\delta_{ij}\eta^{\nu i}\varepsilon^{\rho\mu j\alpha} B_3^{(0,2)}(0) \right)$$

$$+ (\text{symmetric terms}) + \mathcal{O}(Q^2) \tag{4.26}$$

Here $B_3^{(k,l)}(m)$ is defined in Eq. (4.24). As evaluated in the Appendix. A.2, $B_3^{(2,0)}(0)$ and $B_3^{(0,2)}(0)$ are given by

$$B_3^{(2,0)}(0) = \frac{\mu_R^2}{16\pi^2} + \frac{T_0^2}{48} \equiv C(T_0, \mu_R), \quad B_3^{(0,2)}(0) = \frac{3\mu_R^2}{16\pi^2} + \frac{T_0^2}{16} = 3C(T_0, \mu_R). \tag{4.27}$$

As a result, we obtain

$$\begin{aligned}\Psi_2^{(1)}[\lambda] &= \frac{1}{2}\sum_Q 2iC(T_0,\mu_R)\tilde{Q}_\rho\left(\eta^{\nu 0}\varepsilon^{\rho\mu 0\alpha} + \delta_{ij}\eta^{\nu i}\varepsilon^{\rho\mu j\alpha}\right)\delta\tilde{g}_{\mu\nu}(Q)A_\alpha(-Q) \\ &= \int_0^{\beta_0} d\tau \int d^3x \left(\frac{\mu_R^2}{8\pi^2} + \frac{T_0^2}{24}\right)\varepsilon^{0ijk}A_i(x)\partial_j\tilde{g}_{0k}(x)\end{aligned} \tag{4.28}$$

We have evaluated the anomalous part of the Masseiu-Planck functional for the system composed of the right-handed Weyl fermion, which contains at least one external gauge field A_i, and causes the anomaly-induced transport. The result is summarized as follows:

$$\Psi^{(1)}[\lambda] = \int d^3x \frac{\nu_R}{8\pi^2}\varepsilon^{0ijk}A_i\partial_j A_k + \int d^3x \varepsilon^{0ijk}\left(\frac{\nu_R\mu_R}{8\pi^2} + \frac{T_0}{24}\right)A_i(x)\partial_j\tilde{g}_{0k}(x). \tag{4.29}$$

Here we performed the imaginary-time integration since the integrands do not depend on the imaginary time, and introduced $\nu_R \equiv \beta_0\mu_R$. As discussed in Sec. 2.4, the external gauge field is not Kaluza-Klein gauge invariant. In order to see the Kaluza-Klein gauge invariance, we rewrite Eq. (4.29) in terms of the modified gauge field $\tilde{\mathcal{A}}_\mu$ defined in Eq. (2.154), which is manifestly Kaluza-Klein gauge invariant. As a result, we obtain the value of the constant as follows:

$$C = \frac{3}{8\pi^2}, \quad C_3 = \frac{1}{24}, \tag{4.30}$$

The former one is in accordance with the anomaly coefficient [41]. The latter one is also consistent with the one obtained by the use of the Green-Kubo formula [105, 106], or the anomaly inflow mechanism [152].

4.2.3 Anomaly-induced transport from Masseiu-Planck functional

We have evaluated the first-order anomalous correction to the Masseiu-Planck functional (4.29). Then, we can easily extract the anomalous constitutive relation by taking the variation with respect to the external gauge field A_i. It results in

$$\langle \hat{J}_R^i(x)\rangle_{(0,1)}^{\mathrm{LG}} = \frac{1}{\sqrt{-g}}\frac{\delta\Psi^{(1)}}{\delta A_i(x)} = \frac{\mu_R}{4\pi^2}B^i(x) + \left(\frac{\mu_R^2}{8\pi^2} + \frac{T^2}{24}\right)\omega^i(x), \tag{4.31}$$

where we introduced the magnetic field $\boldsymbol{B} \equiv \boldsymbol{\nabla}\times\boldsymbol{A}$, and the vorticity $\boldsymbol{\omega} \equiv \boldsymbol{\nabla}\times\boldsymbol{v}$. Here $(0,1)$ represents the term containing no time derivative, and one spatial derivative, as is introduced in Sec. 3.2. Eq. (4.31) implies that we have the right-handed current along the magnetic field, and fluid vorticity. If we consider the system composed of the left-handed Weyl fermions, by repeating the same procedure, we obtain the left-handed current as follows:

$$\langle \hat{J}_L^i(x)\rangle_{(0,1)}^{\mathrm{LG}} = -\frac{\mu_L}{4\pi^2}B^i(x) - \left(\frac{\mu_L^2}{8\pi^2} + \frac{T^2}{24}\right)\omega^i(x), \tag{4.32}$$

where μ_L denotes the chemical potential for the left-handed Weyl fermion. The difference only appears in the sign of current.

We can consider the system consisting of the massless Dirac fermions as a sum of the right- and left-handed Weyl fermions. Then, recalling the fact that the vector current $\hat{J}_V$ and the axial current are expressed in terms of the right-handed and left-handed current

$$\hat{J}^\mu_V(x) = \hat{J}^\mu_R(x) + \hat{J}^\mu_L(x), \quad \hat{J}^\mu_A(x) = \hat{J}^\mu_R(x) - \hat{J}^\mu_L(x), \tag{4.33}$$

we obtain the constitutive relation of the vector and axial currents. Combining the above results (4.31) and (4.32), we arrive at the following constitutive relations

$$\langle \hat{\boldsymbol{J}}_V(x)\rangle^{\mathrm{LG}}_{(0,1)} = \frac{\mu_5}{2\pi^2}\boldsymbol{B} + \frac{\mu\mu_5}{2\pi^2}\boldsymbol{\omega}, \tag{4.34}$$

$$\langle \hat{\boldsymbol{J}}_A(x)\rangle^{\mathrm{LG}}_{(0,1)} = \frac{\mu}{2\pi^2}\boldsymbol{B} + \left(\frac{\mu^2+\mu_5^2}{4\pi^2} + \frac{T^2}{12}\right)\boldsymbol{\omega}, \tag{4.35}$$

where we introduced the vector chemical potential $\mu = (\mu_R + \mu_L)/2$ and the chiral chemical potential $\mu_5 = (\mu_R - \mu_L)/2$. These constitutive relations correctly describe the chiral magnetic effect, the chiral separation effect, and the chiral vortical effect. In conclusion, together with the leading-order part and the first-order dissipative parts, we have constructed a complete set of the constitutive relations for the system containing the quantum anomaly for the system composed of the Weyl fermions, and for the system composed of the massless Dirac fermions.

4.3 Brief summary

The main results of this chapter can be summarized as follows:

- Considering the system composed of the Weyl fermions, we have perturbatively evaluated the first-order Masseiu-Planck functional (4.29), and derived the anomaly-induced transrport (4.34)-(4.35) (Sec. 4.2).

In this Chapter, the anomaly-induced transport phenomena were derived on the basis of path-integral formulation for local thermal equilibrium developed in Chapter. 2. We first show that our derivation of the hydrodynamic equations for the zeroth-order part and the first-order dissipative part discussed in Chapter. 3 is still robust even in the presence of the quantum anomaly. After that we remarked that the anomaly-induced transport result from the first-ordrer corrections to the Masseiu-Planck functional for parity-violating systems. We, then, show the possible form of the anomalous correction based on the symmetry consideration [41, 86].

On the basis of our original formulation developed in Chapter. 2, we consider the the systems of Weyl fermions, in which the quantum anomaly and the anomaly-induced transport

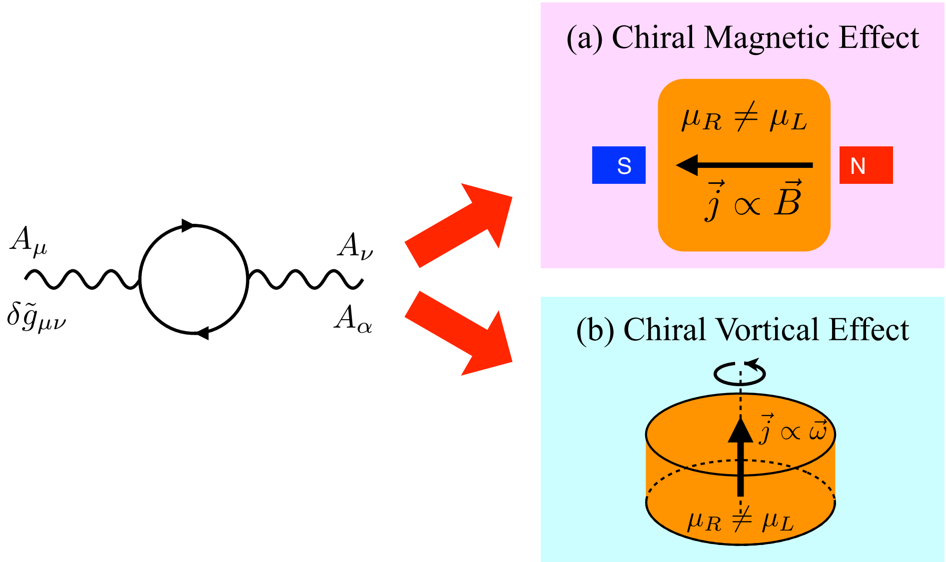


Figure 4.1: Illustrative summary of Chapter 4. From the one-loop calculation of the Masseiu-Planck functional, we obtain the anomalous constitutive relations.

phenomena take place. We perform the perturbative calculation of the Masseiu-Planck functional at one-loop level with the help of the imaginery-time formalism. As a result, we obtain the first-order corrections to the Masseiu-Planck functional. We see that the obtained anomalous constitutive relations correctly describe the chiral magnetic effect, the chiral separation effect, and the chiral vortical effect (see Fig. 4.1).

Chapter 5

Summary and Outlook

In this thesis, we have derived relativistic hydrodynamics from quantum field theories on the basis of the local Gibbs ensemble method. In order to derive the hydrodynamic equations we introduced an assumption that the initial density operator is given by a local Gibbs distribution. This enables us to describe the time evolution of the hydrodynamic variables by approximating the initial density operator by a local Gibbs distribution introduced at later time. We have shown that deviations from the local Gibbs distribution is proportional to the derivatives of hydrodynamic variables, which can be treated in a perturbative way. Then, we have decomposed the constitutive relations for the energy-momentum tensor and charge current into nondissipative and dissipative parts and analyzed their time evolution in detail. While the nondissipative part is completely determined by the local Gibbs distribution at the present time, the dissipative part depends on the past information in general.

First, we focused on the nondissipative part of the constitutive relation, and performed the path-integral formulation of the thermodynamic potential for the local Gibbs distribution. This gives a generalization of the imaginary-time formalism for systems under local thermal equilibrium. We have shown that the thermodynamic pontential, which gives the generating functional of locally thermalized system, is written in terms of the quantum field theory in the curved spacetime. Similar to the standard imaginary-time formalism, this curved spacetime has one imaginary-time direction and $d-1$ spatial directions. The structure of this thermally emergent curved spacetime is completely determined by hydrodynamic variables such as the local temperature, and fluid-four velocity at the present time. The vital point we have demonstrated is that, regardless of the spin of quantum fields, this emergent curved spacetime has the same notable symmetry properties: Kaluza-Klein gauge symmetry, spatial diffeomorphism symmetry, and gauge symmetry. With the help of the symmetry argument, we can write down the general form of the Masseiu-Planck functional. As a result, the nondissipative part of the hydrodynamic equations, in particular the perfect fluid part of the constitutive relation is obtained in the leading-order derivative expansion. Our formalism is also applicable to the situation in the presence of the quantum anomaly, and we have derived the anomaly-induced tranport. In fact, considering the first-order nondissipative corrections in the parity-violating system composed of the chiral fermions, we have obtained the anomaly-induced transport in the nondissipative constitutive relations. With the perturbative calculation, we have evaluated the anomalous transport coefficients at one-loop level.

Second, we have formulated a solid basis to study dissipative corrections in the constitutive relations. In fact, we have obtained the self-consistent equation for the expectation value of the conserved currents which enables us to perform the derivative expansion order-by-order. In particular, by performing the leading order dissipative derivative expansion, together with the result on nondissipative part, we have derived the first-order dissipative hydrodynamic equations, which results in the relativistic version of the Navier-Stokes equation. In addition to the correct constitutive relations, our formalism also gives the quantum field theoretical expression of the Green-Kubo formulas. These results give a complete derivation of the hydrodynamic equation from quantum field theories based on recent developments of nonequilibrium statistical mechanics.

There are several prospects on future research based on our approach. One is an extension to the case in the coexistence of the other zero modes such as the Nambu-Goldstone mode, and the electromagnetic wave. As is mentioned in the introduction, there exists a well-established hydrodynamic way to describe such systems: the superfluid (two-fluid) hydrodynamics, and the magneto-hydrodynamics. Our approach may shed new light on the understanding of the hydrodynamic description of such systems, in particular from the point of the view of quantum field theories and nonequilibrium statistical mechanics. For example, if the underlying quantum field theory has the quantum anomaly, the hydrodynamic equation could be modified in the same way as the (normal) anomalous hydrodynamics discussed in Chapter. 4. It is interesting to derive the chiral superfluid hydrodynamic equation, and the chiral magnetohydrodynamic equation, which would describe the novel transport phenomena related to the quantum anomaly.

Another interesting direction is an application to the second-order hydrodynamic equations. Due to the undesirable acausal property of the first-order relativistic hydrodynamics, there exist a lot of works concerning the derivation of the second-order equations which restore the causality. Although most of them are based on the relativistic Boltzmann equations [23, 24, 25, 26, 27, 28, 29, 30, 31, 32], their analyses are, in principle, only valid in the weakly coupled systems, which is not the case e.g. for the QGP created in heavy-ion collisions. Moreover, an approach based on the fluid/gravity correspondence suggests that there are some nondissipative terms missed in the conventional kinetic approach. In contrast to the kinetic approach, our method is also applicable to strongly coupled systems. Also, it allows us to access the nondissipative part including the anomaly-induced transport through the Masseiu-Planck functional. Therefore, our method gives the way to calculate them based on the imaginary-time formalism. To obtain the second-order hydrodynamic equation with a complete set of transport coefficients is next step to be pursued.

It is of utmost importance to apply the derived hydrodynamic equation to the real physical phenomena such as the QGP created in heavy-ion collisions, the QED plasma and neutrino gases in compact stars, and the emergent relativistic quasi-particle systems in condensed matters. In particular, the applications of anomalous hydrodynamics to the QGP and the neutrino gases have recently been attracted much attentions, although we are still miles away from the complete understanding of the anomaly-induced transport in these situations. Thanks to the microscopic derivation of such transport presented in this thesis, we are now equipped to implement anomalous hydrodynamic simulations which enable us to perform the quantitative analysis of various physical systems.

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Appendix A

Detailed calculation

A.1 Derivation of Eq. (2.47)

Let us, here derive Eq. (2.47). Noting that the volume element can be written as $d\Sigma_{\bar{t}\mu} = d^dx\sqrt{-g}\delta(\bar{t} - \bar{t}(x))\partial_\mu \bar{t}(x) = -d^dx\sqrt{-g}\partial_\mu\theta(\bar{t} - \bar{t}(x))$, we write

$$\begin{aligned}\int d\Sigma_{\bar{t}\mu} f^\mu(x) &= -\int d^dx\sqrt{-g}\partial_\mu\theta(\bar{t} - \bar{t}(x))f^\mu(x) \\ &= \int d^dx\sqrt{-g}\theta(\bar{t} - \bar{t}(x))\nabla_\mu f^\mu(x),\end{aligned} \tag{A.1}$$

where we used the integral by part, and assumed that $f^\mu(x)$ vanishes at the boundary. The derivative of Eq. (A.1) with respect to $\bar{t}$ leads to Eq. (2.47),

$$\begin{aligned}\partial_{\bar{t}}\int d\Sigma_{\bar{t}\mu} f^\mu(x) &= \partial_{\bar{t}}\int d^dx\sqrt{-g}\theta(\bar{t} - \bar{t}(x))\nabla_\mu f^\mu(x) \\ &= \int d\Sigma_{\bar{t}} N(x)\nabla_\mu f^\mu(x),\end{aligned} \tag{A.2}$$

where we used $d\Sigma_{\bar{t}} = d^dx\sqrt{-g}\delta(\bar{t} - \bar{t}(x))N^{-1}$.

A.2 Evaluation of integral (4.24)

Here, we evaluate integrals

$$B_3^{(k,l)}(m) \equiv T_0 \sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{(\tilde{P}_0)^k(\boldsymbol{p})^l}{(\tilde{P}^2 + m^2)^2}. \tag{A.3}$$

in the case of $(l, k) = (1, 0),\ (2, 0),\ (0, 2)$.

Evaluation of $B_3^{(1,0)(0)}$

First, let us evaluate the $B_3^{(1,0)}(0)$. By taking the Mastubara sum, we obtain as follows:

$$
\begin{aligned}
B_3^{(1,0)}(m) &= T\sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\tilde{P}_0}{(\tilde{P}^2+m^2)^2} = -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}T\sum_n \frac{\tilde{P}_0}{\tilde{P}^2+m^2} \\
&= -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}T\sum_n \frac{i\omega_n+\mu}{(i\omega_n+\mu)^2+\boldsymbol{p}^2+m^2} \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\Bigg(\int_{-i\infty+\mu+0^+}^{i\infty+\mu+0^+}\frac{dp_0}{2\pi i}\frac{p_0}{p_0^2+\boldsymbol{p}^2+m^2}\frac{1}{e^{\beta(p_0-\mu)}+1} \\
&\qquad\qquad + \int_{-i\infty+\mu-0^+}^{i\infty+\mu-0^+}\frac{dp_0}{2\pi i}\frac{p_0}{p_0^2+\boldsymbol{p}^2+m^2}\frac{1}{e^{\beta(\mu-p_0)}+1}\Bigg) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\Bigg(\int_{-i\infty+\mu+0^+}^{i\infty+\mu+0^+}\frac{dp_0}{2\pi i}\frac{p_0}{(p_0+\omega_p)(p_0-\omega_p)}\frac{1}{e^{\beta(p_0-\mu)}+1} \\
&\qquad\qquad + \int_{-i\infty+\mu-0^+}^{i\infty+\mu-0^+}\frac{dp_0}{2\pi i}\frac{p_0}{(p_0+\omega_p)(p_0-\omega_p)}\frac{1}{e^{\beta(\mu-p_0)}+1}\Bigg) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\left(-\frac{\omega_p}{2\omega_p}\frac{1}{e^{\beta(\omega_p-\mu)}+1}+\frac{-\omega_p}{-2\omega_p}\frac{1}{e^{\beta(\mu+\omega_p)}+1}\right) \\
&= \frac{1}{2}\frac{1}{2\pi^2}\int_0^\infty p^2 dp \frac{1}{2p}\frac{\partial}{\partial p}\left(-\frac{1}{e^{\beta(\omega_p-\mu)}+1}+\frac{1}{e^{\beta(\mu+\omega_p)}+1}\right) \\
&= -\frac{1}{8\pi^2}\int_0^\infty dp\left(-\frac{1}{e^{\beta(\omega_p-\mu)}+1}+\frac{1}{e^{\beta(\mu+\omega_p)}+1}\right),
\end{aligned}
\tag{A.4}
$$

Here, we drop terms which do not depend on the thermodynamic variables. Then, putting $m = 0$, we obtain

$$
\begin{aligned}
B_3^{(1,0)}(m=0) &= -\frac{1}{8\pi^2}\int_0^\infty dp\left(\frac{1}{e^{\beta(p+\mu)}+1}-\frac{1}{e^{\beta(p-\mu)}+1}\right) \\
&= -\frac{1}{8\pi^2\beta}\int_0^\infty dx\left(\frac{1}{e^{x+\nu}+1}-\frac{1}{e^{x-\nu}+1}\right) \\
&= -\frac{1}{8\pi^2\beta}\left(\log(1+e^{-\nu})-\log(1+e^{\nu})\right) \\
&= \frac{\nu}{8\pi^2\beta} = \frac{\mu}{8\pi^2}.
\end{aligned}
\tag{A.5}
$$

Evaluation of $B_3^{(2,0)(0)}$

Next, let us evaluate the $B_3^{(2,0)}(0)$ in a similar way as $B^{(1,0)(0)}$.

$$\begin{aligned}
B_3^{(2,0)}(m) &= T\sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{(\tilde{P}_0)^2}{(\tilde{P}^2+m^2)^2} = -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}T\sum_n \frac{(\tilde{P}_0)^2}{\tilde{P}^2+m^2} \\
&= -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}T\sum_n \frac{(i\omega_n+\mu)^2}{(i\omega_n+\mu)^2+\boldsymbol{p}^2+m^2} \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\Bigg(\int_{-i\infty+\mu+0^+}^{i\infty+\mu+0^+}\frac{dp_0}{2\pi i}\frac{p_0^2}{p_0^2+\boldsymbol{p}^2+m^2}\frac{1}{e^{\beta(p_0-\mu)}+1} \\
&\qquad\qquad + \int_{-i\infty+\mu-0^+}^{i\infty+\mu-0^+}\frac{dp_0}{2\pi i}\frac{p_0^2}{p_0^2+\boldsymbol{p}^2+m^2}\frac{1}{e^{\beta(\mu-p_0)}+1}\Bigg) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\Bigg(\int_{-i\infty+\mu+0^+}^{i\infty+\mu+0^+}\frac{dp_0}{2\pi i}\frac{p_0^2}{(p_0+\omega_p)(p_0-\omega_p)}\frac{1}{e^{\beta(p_0-\mu)}+1} \\
&\qquad\qquad + \int_{-i\infty+\mu-0^+}^{i\infty+\mu-0^+}\frac{dp_0}{2\pi i}\frac{p_0^2}{(p_0+\omega_p)(p_0-\omega_p)}\frac{1}{e^{\beta(\mu-p_0)}+1}\Bigg) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\left(-\frac{\omega_p^2}{2\omega_p}\frac{1}{e^{\beta(\omega_p-\mu)}+1}+\frac{\omega_p^2}{-2\omega_p}\frac{1}{e^{\beta(\mu+\omega_p)}+1}\right) \\
&= -\frac{1}{2}\frac{1}{2\pi^2}\int_0^\infty p^2dp\frac{1}{2p}\frac{\partial}{\partial p}\left(\frac{\omega_p}{e^{\beta(\omega_p-\mu)}+1}+\frac{\omega_p}{e^{\beta(\mu+\omega_p)}+1}\right) \\
&= \frac{1}{8\pi^2}\int_0^\infty dp\left(\frac{\omega_p}{e^{\beta(\omega_p-\mu)}+1}+\frac{\omega_p}{e^{\beta(\mu+\omega_p)}+1}\right),
\end{aligned} \tag{A.6}$$

Here, we again drop terms which do not depend on the thermodynamic variables. Putting $m=0$, we obtain

$$\begin{aligned}
B_3^{(2,0)}(m=0) &= \frac{1}{8\pi^2}\int_0^\infty dp\left(\frac{p}{e^{\beta(p-\mu)}+1}+\frac{p}{e^{\beta(p+\mu)}+1}\right) \\
&= \frac{1}{8\pi^2\beta^2}\int_0^\infty dx\left(\frac{x}{e^{x-\nu}+1}+\frac{x}{e^{x+\nu}+1}\right) \\
&= \frac{\mu^2}{16\pi^2}+\frac{T^2}{48} \equiv C(\mu,T).
\end{aligned} \tag{A.7}$$

Evaluation of $B_3^{(0,2)}(0)$

Let us evaluate the $B_3^{(0,2)}(0)$.

$$
\begin{aligned}
B_3^{(0,2)}(m) &= T\sum_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\boldsymbol{p}^2}{(\tilde{P}^2+m^2)^2} = -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}T\sum_n \frac{\boldsymbol{p}^2}{\tilde{P}^2+m^2} \\
&= -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}T\sum_n \frac{\boldsymbol{p}^2}{(i\omega_n+\mu)^2+\boldsymbol{p}^2+m^2} \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\left(\int_{-i\infty+\mu+0^+}^{i\infty+\mu+0^+}\frac{dp_0}{2\pi i}\frac{\boldsymbol{p}^2}{p_0^2+\boldsymbol{p}^2+m^2}\frac{1}{e^{\beta(p_0-\mu)}+1}\right. \\
&\qquad\qquad \left. + \int_{-i\infty+\mu-0^+}^{i\infty+\mu-0^+}\frac{dp_0}{2\pi i}\frac{\boldsymbol{p}^2}{p_0^2+\boldsymbol{p}^2+m^2}\frac{1}{e^{\beta(\mu-p_0)}+1}\right) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial}{\partial m^2}\left(\int_{-i\infty+\mu+0^+}^{i\infty+\mu+0^+}\frac{dp_0}{2\pi i}\frac{\boldsymbol{p}^2}{(p_0+\omega_p)(p_0-\omega_p)}\frac{1}{e^{\beta(p_0-\mu)}+1}\right. \\
&\qquad\qquad \left. + \int_{-i\infty+\mu-0^+}^{i\infty+\mu-0^+}\frac{dp_0}{2\pi i}\frac{\boldsymbol{p}^2}{(p_0+\omega_p)(p_0-\omega_p)}\frac{1}{e^{\beta(\mu-p_0)}+1}\right) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\boldsymbol{p}^2\frac{\partial}{\partial m^2}\left(-\frac{1}{2\omega_p}\frac{1}{e^{\beta(\omega_p-\mu)}+1}+\frac{1}{-2\omega_p}\frac{1}{e^{\beta(\mu+\omega_p)}+1}\right) \\
&= -\frac{1}{2}\frac{1}{2\pi^2}\int_0^\infty p^4 dp\frac{1}{2p}\frac{\partial}{\partial p}\left(\frac{\boldsymbol{p}^2/\omega_p}{e^{\beta(\omega_p-\mu)}+1}+\frac{\boldsymbol{p}^2/\omega_p}{e^{\beta(\mu+\omega_p)}+1}\right) \\
&= \frac{3}{8\pi^2}\int_0^\infty dp\frac{p^2}{\omega_p}\left(\frac{1}{e^{\beta(\omega_p-\mu)}+1}+\frac{1}{e^{\beta(\mu+\omega_p)}+1}\right),
\end{aligned} \tag{A.8}
$$

Here, we drop terms which do not depend on the thermodynamic variables. Putting $m=0$, we obtain

$$
\begin{aligned}
B_3^{(0,2)}(m=0) &= \frac{3}{8\pi^2}\int_0^\infty dp\left(\frac{p}{e^{\beta(p-\mu)}+1}+\frac{p}{e^{\beta(p+\mu)}+1}\right) \\
&= \frac{3}{8\pi^2\beta^2}\int_0^\infty dx\left(\frac{x}{e^{x-\nu}+1}+\frac{x}{e^{x+\nu}+1}\right) \\
&= \frac{3\mu^2}{16\pi^2}+\frac{T^2}{16} = 3C(T,\mu).
\end{aligned} \tag{A.9}
$$

Appendix B

Ambiguity of energy-momentum tensor

As is well known, there is an ambiguity for the definition of the energy-momentum tensor. As we see from now on, a part of ambiguity is related to the definition of an angular momentum density of the system. Here, we restrict ourselves to discussion in the flat spacetime for simplicity.

The invariance of the Lagrangian under the space-time translations and the Lorentz transformation, together with Noether's theorem, tells us the existence of a so-called canonical energy-momentum tensor $\Theta^{\mu}{}_{\nu}$ and a canonical angular momentum density tensor $M^{\mu}{}_{\rho\sigma}$

$$\Theta^{\mu}{}_{\nu} = -\frac{1}{2}\bar{\psi}(\gamma^{\mu}\overrightarrow{\partial}_{\nu} - \overleftarrow{\partial}_{\nu}\gamma^{\mu})\psi - \delta^{\mu}_{\nu}\mathcal{L}, \tag{B.1}$$

$$M^{\mu}{}_{\rho\sigma} = M^{\mu(\text{orbital})}_{\rho\sigma} + M^{\mu(\text{spin})}_{\rho\sigma}, \tag{B.2}$$

where the canonical angular momentum density tensor is composed of an orbital angular momentum density $M^{\mu(\text{orbital})}_{\rho\sigma}$ and a spin density $M^{\mu(\text{spin})}_{\rho\sigma}$, which are given by

$$M^{\mu(\text{orbital})}_{\rho\sigma} = x_{\rho}\Theta^{\mu}{}_{\sigma} - x_{\sigma}\Theta^{\mu}{}_{\rho}, \tag{B.3}$$

$$M^{\mu(\text{spin})}_{\rho\sigma} = \frac{i}{2}\bar{\psi}\{\gamma^{\mu}, \Sigma_{\rho\sigma}\}\psi. \tag{B.4}$$

If we raise the subscript $\Theta^{\mu\nu} = g^{\nu\rho}\Theta^{\mu}{}_{\rho}$, the canonical energy-momentum tensor $\Theta^{\mu\nu}$ is not symmetric under $\mu \leftrightarrow \nu$. Furthermore, it is also not gauge invariant in general.

On the other hand, using the canonical energy-momentum tensor $\Theta^{\mu\nu}$, we can construct a symmetric and gauge-invariant energy-momentum tensor, which is so-called a Belinfante energy-momentum tensor $T^{\mu\nu}$ given by

$$\begin{aligned} T^{\mu\nu} &= \Theta^{\mu\nu} + \partial_{\rho}G^{\rho\mu\nu}, \\ &= -\delta^{\mu}_{\nu}\mathcal{L} - \frac{1}{4}\bar{\psi}(\gamma^{\mu}\overrightarrow{D}_{\nu} + \gamma_{\nu}\overrightarrow{D}^{\mu} - \overleftarrow{D}_{\nu}\gamma^{\mu} - \overleftarrow{D}^{\mu}\gamma_{\nu})\psi \end{aligned} \tag{B.5}$$

where $G^{\rho\mu\nu}$ is defined using the spin density $M^{\mu\rho\sigma}_{\text{spin}}$ as

$$G^{\rho\mu\nu} \equiv \frac{1}{2}\left(M^{\rho\mu\nu}_{(\text{spin})} + M^{\mu\nu\rho}_{(\text{spin})} + M^{\nu\mu\rho}_{(\text{spin})}\right). \tag{B.6}$$

Since $G^{\rho\mu\nu}$ is anti-symmetric with respect to its first indices $G^{\rho\mu\nu} = -G^{\mu\rho\nu}$, the Belinfante energy-momentum tensor also satisfied the conservation law:

$$\partial_\mu T^{\mu\nu} = \partial_\mu \Theta^{\mu\nu} + \partial_\mu \partial_\rho G^{\rho\mu\nu} = 0. \tag{B.7}$$

It is also important to note that $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ give the same total energy-momentum

$$\int d^3x T^{0\nu} - \int d^3x \Theta^{0\nu} = \int d^3x \partial_\rho G^{\rho 0\nu} = \int d^3x \partial_i G^{i0\nu} = 0, \tag{B.8}$$

where we assume that the fields fall off sufficiently rapidly as $|\boldsymbol{x}| \to \infty$ to obtain the final result.

In a similar way, we can define a Belinfante angular momentum density tensor $J^{\mu\rho\sigma}$ as

$$\begin{aligned} J^{\mu\rho\sigma} &= M^{\mu\rho\sigma} + \partial_\lambda \left(x^\rho G^{\lambda\mu\sigma} - x^\sigma G^{\lambda\mu\rho} \right), \\ &= x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}, \end{aligned} \tag{B.9}$$

where we used the relation $M^{\mu\rho\sigma}_{\text{spin}} = G^{\sigma\mu\rho} - G^{\rho\mu\sigma}$ which follows from the definition of $G^{\rho\mu\nu}$ in Eq. (B.6) to obtain the second line. The Belinfante angular momentum density has, therefore, the form of a purely orbital angular momentum tensor associated with the Belinfante energy-momentum tensor. It is trivially conserved as a consequence of the symmetry of $T^{\mu\nu}$ under $\mu \leftrightarrow \nu$.

From above two properties, we usually do not mind the difference between $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ and freely choose a convenient one. However, as discussed in Sec. 2.2.2, the one coupled to the metric in the curved spacetime is symmetric energy-momentum tensor. In order to formulate the imaginary-time formalism in local thermal equilibrium, we have to adopt the symmetric one. Otherwise, it turns out that we have to add another term resulted from the angular momentum.

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