TOPOLOGICAL DIPOLE SYMMETRIES

Based on : TB, Yamamoto, Yokokura
SciPost Phys. 16 (2024) 051

INTRODUCTION : DIPOLE SYMMETRY

Standard conservation law : $\partial_\mu J^\mu = \partial_0 J^0 + \partial_i J^i = 0$

$$\Big\downarrow \text{ what if } J^i = \partial_j J^{ij}\,?$$

Dipole conservation law : $\partial_0 J^0 + \partial_i \partial_j J^{ij} = 0$

<u>Example</u> : theory of a Lifshitz scalar, $\mathcal{L} = \frac{1}{2}(\partial_0 \phi)^2 - \frac{1}{2}(\partial_i \partial_j \phi)^2$

invariance under $\phi \to \phi + \epsilon \implies J^0 = \partial_0 \phi\,,\ J^{ij} = \partial_i \partial_j \phi$

$$\partial_0 J^0 + \partial_i \partial_j J^{ij} = 0$$

conserved scalar charge

$$Q \equiv \int d^d\vec{x}\; J^0(\vec{x},t)$$

conserved dipole moment

$$D^i \equiv \int d^d\vec{x}\; x^i J^0(\vec{x},t)$$

$$\partial_0 J^0 + \partial_i \partial_j J^{ij} = 0$$

conserved scalar cherge

$$Q \equiv \int d^d\vec{x}\; J^0(\vec{x},t)$$

conserved dipole moment

$$D^i \equiv \int d^d\vec{x}\; x^i J^0(\vec{x},t)$$

if $\delta_{ij} J^{ij} = 0$

conserved trace of quadrupole moment

$$X \equiv \int d^d\vec{x}\; \vec{x}^2 J^0(\vec{x},t)$$

$$\partial_0 J^0 + \partial_i \partial_j J^{ij} = 0$$

conserved scalar charge

$$Q \equiv \int d^d\vec{x}\; J^0(\vec{x},t)$$

conserved dipole moment

$$D^i \equiv \int d^d\vec{x}\; x^i J^0(\vec{x},t)$$

If Q is finite and nonzero, then D^i/Q is the center of charge.

Conservation of Q & D^i $\Longrightarrow$ mobility constraint!

MAIN RESULTS

① Universality of the local dipole conservation law, in 2+1 dim:

- $J^0 \longleftrightarrow$ symplectic form of the system
- $J^{ij} \longleftrightarrow$ stress tensor

② Extended momentum algebra, in 2+1 dim:

momentum $P_i = -\epsilon_{ij} D^j \quad \Rightarrow \quad \{P_i, P_j\} = -\epsilon_{ij} Q$

③ Absence of well-defined momentum density

aka "linear momentum problem" $\Rightarrow$ constraints on the low-energy spectrum

LOCAL DIPOLE CONSERVATION LAWS

<u>Starting point</u> : phase-space formulation of bosonic field theory

- phase space : set of maps $\phi^A : \mathbb{R}^d \to M$ (target space)
- action functional :

$$S = \int d^d\vec{x}\, dt \left(\omega_A(\phi)\, \partial_0 \phi^A - \mathcal{H}[\phi] \right)$$

symplectic potential

1-form on M : $\omega(\phi) \equiv \omega_A(\phi)\, d\phi^A$

Hamiltonian density

depends on ϕ^A, $\partial_i \phi^A$, ...

<u>Assumptions</u> :
- invariance under spacetime translations
- no higher-order temporal or mixed derivatives

Follow Noether-like reasoning: perform variation under an infinitesimal local spatial translation $x^i \to x^i + \xi^i(\vec{x},t)$:

$$\delta_\xi \int d^d\vec{x}\, \mathcal{H} = \int d^d\vec{x}\, \frac{1}{2}\left(\partial^i \xi^j + \partial^j \xi^i\right) \sigma_{ij}$$

stress tensor

$$\Big\downarrow$$

pull-back of ω to spacetime by ϕ^A :

$$\delta_\xi S = \int d^d\vec{x}\, dt\, \xi^i \left(\partial_0 \omega_i - \partial_i \omega_0 + \partial_j \sigma^j{}_i\right) \qquad \omega_\mu \equiv \omega_A \partial_\mu \phi^A$$

$$\Big\downarrow \text{ for on-shell fields}$$

$$\partial_0 \omega_i - \partial_i \omega_0 = -\partial_j \sigma^j{}_i$$

(local momentum conservation in disguise)

Closed symplectic form $\Omega \equiv d\omega$ $\Rightarrow$ topological current

$$J^{\mu_1\cdots\mu_{d-1}} \equiv \epsilon^{\mu_1\cdots\mu_{d-1}\nu\lambda}\partial_\nu\omega_\lambda$$

Special case of $d=2$ spatial dimensions:

$$J^0 \equiv \rho = \epsilon^{ij}\partial_i\omega_j$$

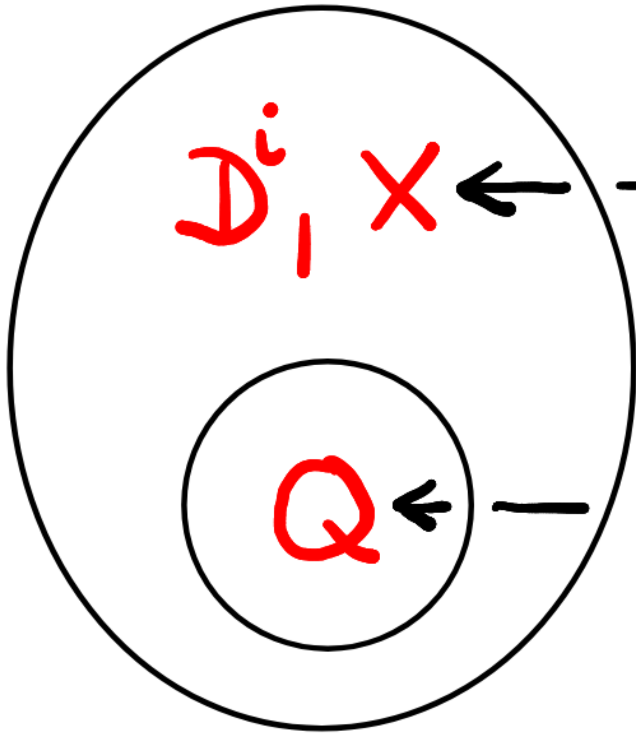
pull-back of Ω to $\mathbb{R}^2$ (space)

$$J^i = \epsilon^{ij}(\partial_j\omega_0 - \partial_0\omega_j) = \partial_j(\epsilon^{ik}\sigma^j{}_k)$$

$$J^{ij} \equiv \frac{1}{2}(\epsilon^{ik}\sigma^j{}_k + \epsilon^{jk}\sigma^i{}_k)$$

$$\partial_0\rho + \partial_i\partial_j J^{ij} = 0$$

This describes nested conservation laws:



on-shell … momentum & angular momentum

off-shell … new topological charge (generalized vorticity)

$$Q \equiv \int d^2\vec{x}\, \rho(\vec{x},t)$$

$$D^i \equiv \int d^2\vec{x}\, x^i \rho(\vec{x},t)$$

$$X \equiv \int d^2\vec{x}\, \vec{x}^2 \rho(\vec{x},t)$$

$$J^{ij} = \frac{1}{2}\left(\epsilon^{ik}\sigma^{j}{}_{k} + \epsilon^{jk}\sigma^{i}{}_{k}\right)$$

J^{ij} is traceless $\Leftrightarrow$ σ^{ij} is symmetric

LIE ALGEBRA OF SPATIAL SYMMETRIES

Poisson bracket of functionals on phase space from symplectic structure:

$$\{F,G\} \equiv \int d^d\vec{x}\ \Omega^{AB}(\phi(\vec{x}))\ \frac{\delta F}{\delta\phi^A(\vec{x})}\ \frac{\delta G}{\delta\phi^B(\vec{x})}$$

matrix inverse of $\Omega_{AB} \equiv \frac{\partial\omega_B}{\partial\phi^A} - \frac{\partial\omega_A}{\partial\phi^B}$

Q is topological charge $\Longleftrightarrow$ all Poisson brackets of Q vanish.

Working in d=2 dimensions, introduce a class of deformed charges:

$$Q_\lambda \equiv \int d^2\vec{x}\ \lambda(\vec{x})\,\epsilon^{ij}\partial_i\omega_j(\phi(\vec{x})) = \frac{1}{2}\int d^2\vec{x}\ \lambda(\vec{x})\,\epsilon^{ij}\,\Omega_{AB}(\phi(\vec{x}))\,\partial_i\phi^A(\vec{x})\,\partial_j\phi^B(\vec{x})$$

$$Q_\lambda \equiv \int d^2\vec{x}\; \lambda(\vec{x})\, \epsilon^{ij} \partial_i \omega_j(\phi(\vec{x})) = \frac{1}{2}\int d^2\vec{x}\; \lambda(\vec{x})\, \epsilon^{ij}\, \Omega_{AB}(\phi(\vec{x}))\, \partial_i\phi^A(\vec{x})\, \partial_j\phi^B(\vec{x})$$

The set Q_λ generates volume-preserving spatial diffeomorphisms :

$$\delta_\lambda \phi^A(\vec{x}) \equiv \{\phi^A(\vec{x}), Q_\lambda\} = -\epsilon^{ij}\partial_i\lambda(\vec{x})\,\partial_j\phi^A(\vec{x}) \quad \Longrightarrow \quad \{Q_\lambda, Q_{\bar{\lambda}}\} = -Q_{\{\lambda,\bar{\lambda}\}}$$

"classical w_∞-algebra"

Subset of Q_λ generates spatial translations & rotations :

$$\left.\begin{aligned} P_i &= Q_{-\epsilon_{ij}x^j} = -\epsilon_{ij} D^j \\ L &= Q_{\vec{x}^2/2} = \frac{1}{2} X \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} &\{L, P_i\} = \epsilon_i{}^j P_j \\ &\boxed{\{P_i, P_j\} = -\epsilon_{ij} Q} \end{aligned}$$

EXAMPLES

FERROMAGNETS

- Target space : $M = SU(2)/U(1) \cong S^2$
- Symplectic structure fixed by $\{n^a(\vec{x}), n^b(\vec{y})\} = \frac{1}{m}\epsilon^{ab}{}_c\, n^c(\vec{x})\, \delta(\vec{x}-\vec{y})$

magnetization density

$$\Updownarrow$$

$$\Omega = -\frac{m}{2}\epsilon_{abc}\, n^a\, dn^b \wedge dn^c$$

- Topological charge : winding number of $\vec{n} : \mathbb{R}^2 \to S^2$

$$Q[\vec{n}] = -\frac{m}{2}\int d^2\vec{x}\, \epsilon^{ij}\, \vec{n}\cdot(\partial_i\vec{n}\times\partial_j\vec{n}) = -4\pi m\, w[\vec{n}]$$

$$\Downarrow$$

$$\boxed{\{P_i, P_j\} = 4\pi\,\epsilon_{ij}\, m\, w[\vec{n}]}$$

SUPERFLUIDS

- EFT of Gross-Pitaevskii type : $\mathcal{M} = \mathbb{C}$
- Symplectic structure : $S = \int d^2\vec{x}\,dt\; i\psi^\dagger \partial_0 \psi + \dots$

$$\Downarrow$$

$$\omega = i\psi^\dagger d\psi \quad \Rightarrow \quad \Omega = i\, d\psi^\dagger \wedge d\psi$$

- Topological charge

$$Q[\psi] = i\int d^2\vec{x}\; \epsilon^{ij}\partial_i\psi^\dagger \partial_j\psi \xrightarrow[\psi(\vec{x},t)\,\longrightarrow\,\sqrt{n_0}\,e^{i\theta(\vec{x},t)}]{\text{vortex solution}} -2\pi n_0\, w[\theta]$$

winding number of the superfluid phase

$$\boxed{\{P_i, P_j\} = 2\pi\epsilon_{ij}\, n_0\, w[\theta]}$$

Where does the nonzero topological charge come from?

system	symplectic form Ω	configuration with $Q \neq 0$	
ferromagnet	closed but not exact	skyrmion	$H^2_{dR}(S^2)$
superfluid	exact	vortex	$H^1_{dR}(S^1)$

LINEAR MOMENTUM PROBLEM

How can $\{P_i, P_j\}$ ever be nonzero? Start from the local momentum algebra satisfied by momentum density $p_i(\vec{x})$:

$$\delta_i p_j(\vec{x}) \equiv \{p_j(\vec{x}), P_i\} = -\partial_i p_j(\vec{x})$$

integrate over space

Still nonzero result?

fields with singularities

fields with nontrivial asymptotics at infinity

momentum density does not exist!

What's wrong with $p_i = -\omega_A \partial_i \phi^A = -\omega_i$ (Noether)?

- satisfies a local conservation law ✓
- not defined globally on M if Ω is closed but not exact ✗

What's wrong with $\tilde{p}_i = -\epsilon_{ij} x^j \underbrace{\epsilon^{kl} \partial_k \omega_l}_{= \rho}$ (dipole moment of topological charge)?

- well-defined globally on M ✓
- violates the local momentum algebra ✗

$$\{\tilde{p}_j(\vec{x}), \tilde{P}_i\} = -\partial_i \tilde{p}_j(\vec{x}) + \epsilon_{ij}\, \rho(\vec{x})$$

What's wrong with $T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$ or equivalent?

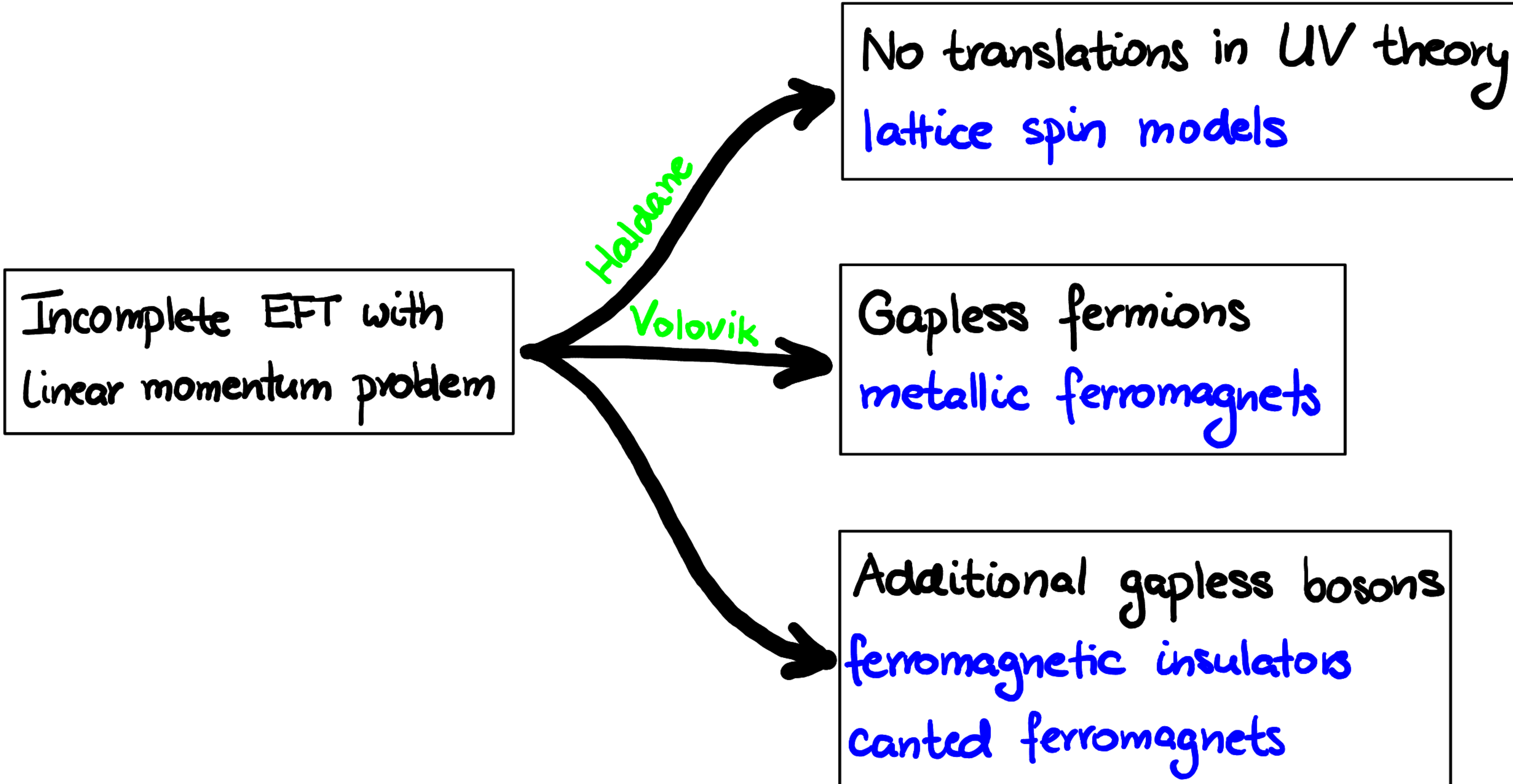
Absence of well-defined momentum density $\Rightarrow$ consistent coupling to background geometry not possible.

Linear momentum problem in the IR

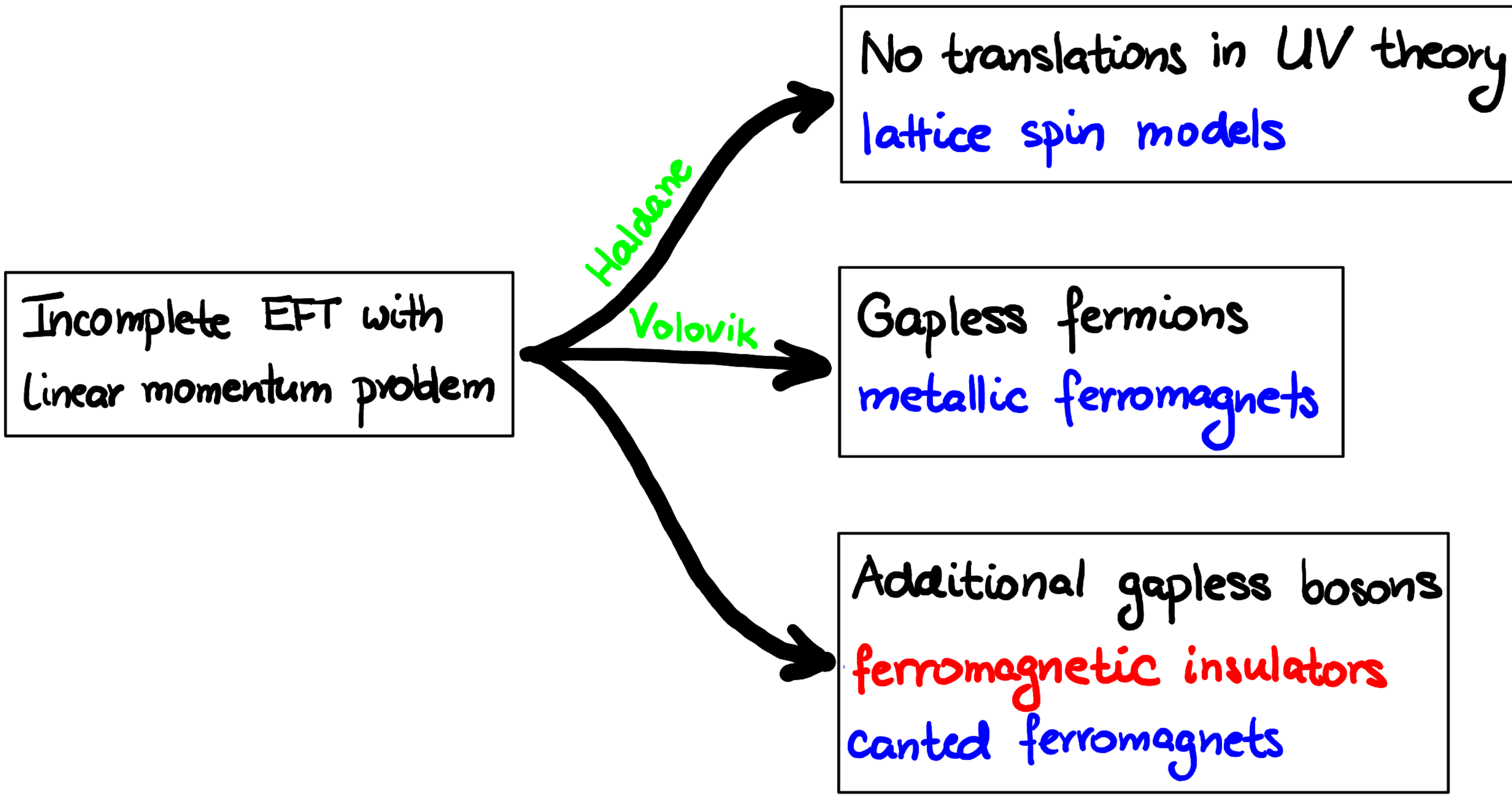
Translationally-invariant physics in the UV

$\Rightarrow$ The low-energy EFT is incomplete!

FERROMAGNETS



FERROMAGNETS



CURING LMP BY CLASSICAL MEDIUM

Lagrangian description of classical matter : $\vec{x} \longrightarrow X^a(\vec{x},t)$

material coordinates

Matter current : $J^\mu \equiv \frac{n_0}{d!}\epsilon^{\mu\nu_1\dots\nu_d}\epsilon_{a_1\dots a_d}\partial_{\nu_1}X^{a_1}\cdots\partial_{\nu_d}X^{a_d} \leftrightarrow * dX^1\wedge\dots\wedge dX^d$

Action in presence of the medium : medium velocity

$$S = \int d^d\vec{x}\,dt\,\omega_A\partial_0\phi^A + \dots \longrightarrow \int d^d\vec{x}\,dt\,\det\{\partial_i X^a\}\,\omega_A(\partial_0+\vec{v}\cdot\vec{\nabla})\phi^A + \dots$$

$$= \frac{1}{d!}\int \epsilon_{a_1\dots a_d}\,\omega(\phi)\wedge dX^{a_1}\wedge\dots\wedge dX^{a_d}$$

The entire momentum is now carried by the medium variables X^a.

Conclusions

1. New type of dipole symmetry where:
   - Q is a topological charge
   - spatial momentum $\Leftrightarrow$ dipole moment of Q
2. Streamlined derivation of extended momentum algebra $\{P_i, P_j\} = -\epsilon_{ij} Q$
   - descends directly from the symplectic structure
   - robust against perturbations
3. New insight in the linear momentum problem:
   - general constraint on the IR spectrum ("classical anomaly")
   - physical consequences of the predicted magnon-phonon coupling?

BACKUP

GENERALIZATION TO HIGHER DIMENSIONS

LOCAL CONSERVATION LAW

$$J^{\mu_1\cdots\mu_{a-1}} \equiv \epsilon^{\mu_1\cdots\mu_{a-1}\nu\lambda}\partial_\nu\omega_\lambda$$

$$\Big\downarrow \quad \partial_0\omega_i - \partial_i\omega_0 = -\partial_j\sigma^j{}_i$$

Dipole-type conservation law:

$$\partial_0\rho^{i_1\cdots i_{a-2}} + \partial_j\partial_k J^{i_1\cdots i_{a-2}jk} = 0$$

tensor charge density

$$\rho^{i_1\cdots i_{a-2}} = \epsilon^{i_1\cdots i_{a-2}jk}\partial_j\omega_k$$

tensor current

$$J^{i_1\cdots i_{a-2}jk} = \frac{1}{2}\left(\epsilon^{i_1\cdots i_{a-2}jl}\sigma^k{}_l + \epsilon^{i_1\cdots i_{a-2}kl}\sigma^j{}_l\right)$$

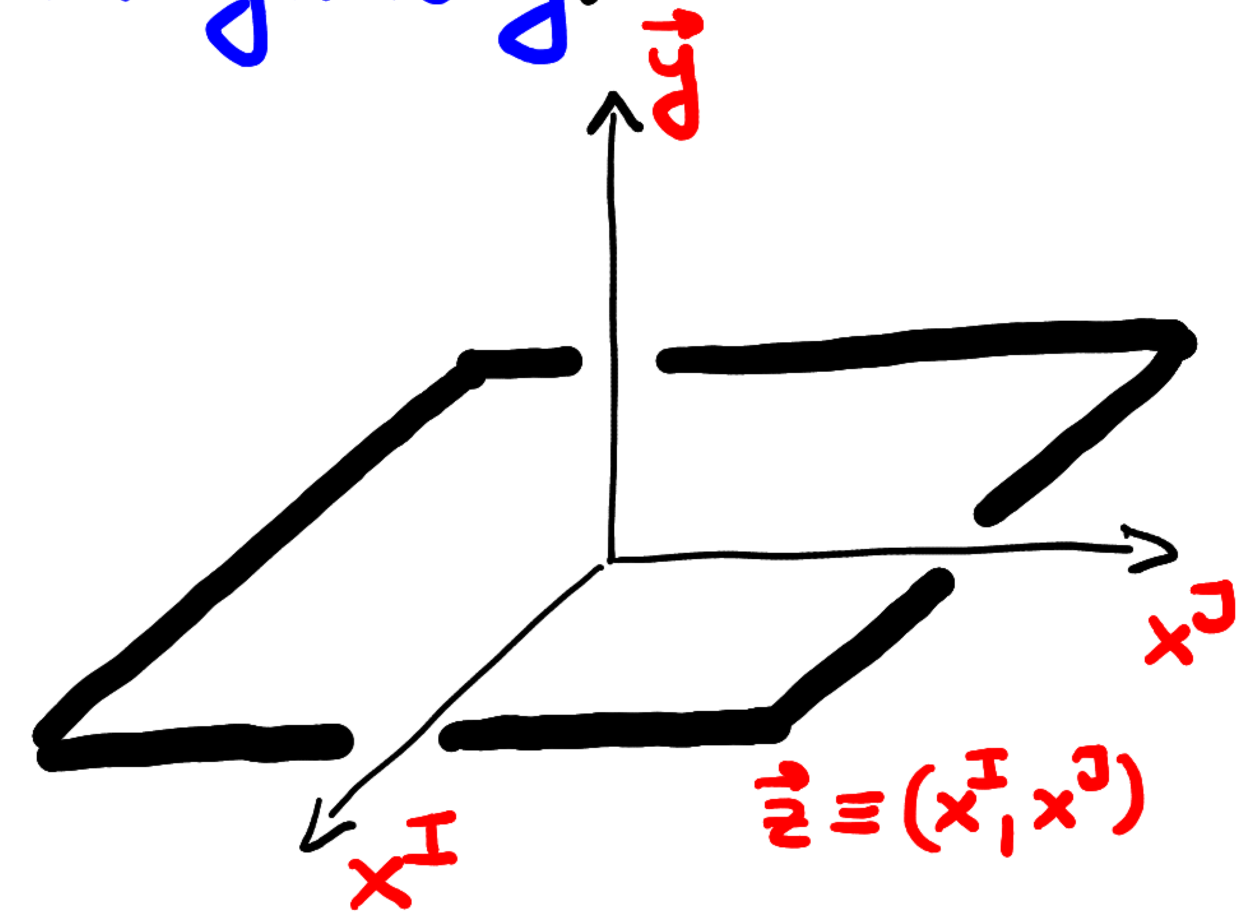
LIE ALGEBRA OF MOMENTUM

The closed 2-form Ω defines a (d−2)-form symmetry.

Define generalized charges by integration over a selected coordinate plane:

$$Q_\lambda(\vec{y}) \equiv \int d^2\vec{z}\; \lambda(\vec{x})\, \epsilon^{ij}\, \partial_i \omega_j(\phi(\vec{x}))$$

run over $\{I, J\}$



These generate in-plane transformations of local fields:

$$\delta_{\lambda,\vec{y}'}\phi^A(\vec{x}) \equiv \{\phi^A(\vec{x}), Q_\lambda(\vec{y}')\} = -\,\epsilon^{ij}\,\partial_i\lambda(\vec{x})\,\partial_j\phi^A(\vec{x})\,\delta(\vec{y}-\vec{y}')$$

Generalized algebra of in-plane generators :

$$\{Q_\lambda(\vec{y}), Q_{\bar{\lambda}}(\vec{y}')\} = -Q_{\{\lambda,\bar{\lambda}\}}(\vec{y})\,\delta(\vec{y}-\vec{y}')$$

$$\{L(\vec{y}), P_i(\vec{y}')\} = \epsilon_i{}^j P_j(\vec{y})\,\delta(\vec{y}-\vec{y}')$$

$$\{P_i(\vec{y}), P_j(\vec{y}')\} = -\epsilon_{ij}\, Q_1(\vec{y})\,\delta(\vec{y}-\vec{y}')$$

topological charge measured in the chosen coordinate plane

integral momentum

$P_i \equiv \int d^{d-2}\vec{y}\; P_i(\vec{y})$

$$\boxed{\{P_i, P_j(\vec{y})\} = -\epsilon_{ij} Q_1(\vec{y})}$$

avoids the no-go theorem on central extensions of Euclidean algebra in $d>2$ dim

LINEAR MOMENTUM PROBLEM FOR HIGHER-FORM SYMMETRIES

Maxwell's electrodynamics coupled to axion background:

$$\mathcal{L} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + c\theta\,\vec{E}\cdot\vec{B}$$

LMP: impossible to define consistent momentum density even when θ has a constant gradient

- $\vec{p} = \vec{E}\times\vec{B}$ ... gauge-invariant ✓
  not locally conserved ✗
- $\vec{p}_\theta = \vec{E}\times\vec{B} + \frac{c}{2}\vec{\nabla}\theta\,(\vec{A}\cdot\vec{B})$ ... locally conserved ✓
  not gauge-invariant ✗

Same constraints on UV completion as in ferromagnets!

Electrodynamics coupled to axion background

No translations

fixed background $\Theta(\vec{x})$

Gapless fermions

meson supercurrent phase of dense QCD

Gapless bosons

chiral soliton lattice