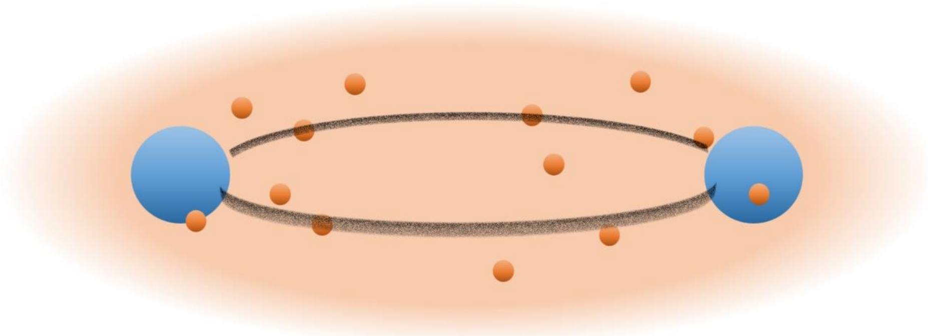
Complex-valued potential between heavy quarkonia in a thermal pion gas



Shota Kaneko
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Based on the collaboration with Masaru Hongo

~Summary~

- ✓ The potential between heavy quarkonia in a thermal pion gas has an imaginary part due to interactions with thermal pions.
- ✓ The imaginary part of the potential is found to show r^{-2} behavior at long distances.

~Motivation~

✓ **Low-energy** hadron-hadron interactions

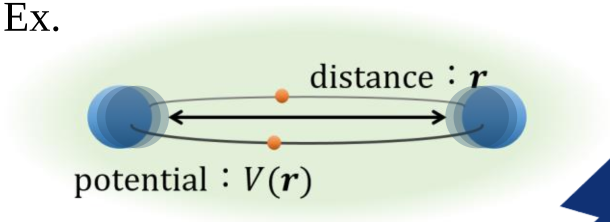
→ Exotic hadrons, nuclear many-body systems, etc.

✓ Recent developments **at zero temperature**

→ Numerical calculations Ex. HAL-QCL method

→ **Model calculations**

Ex.

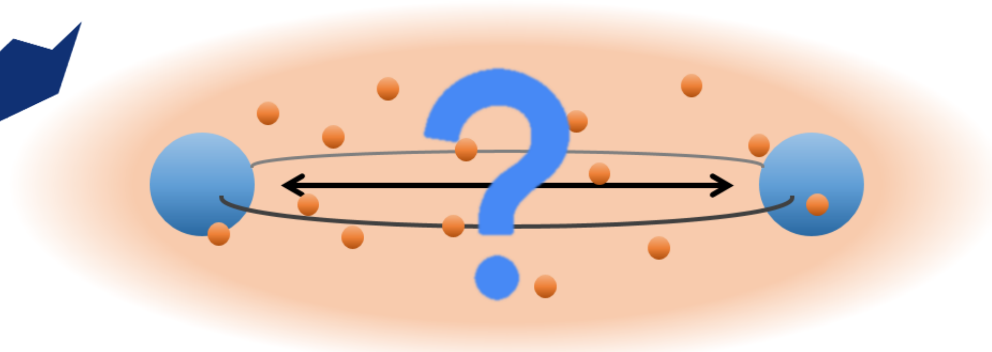


[Nora Brambilla et al., Phys. Rev. D **93**, 054002 (2016).]

● : pion

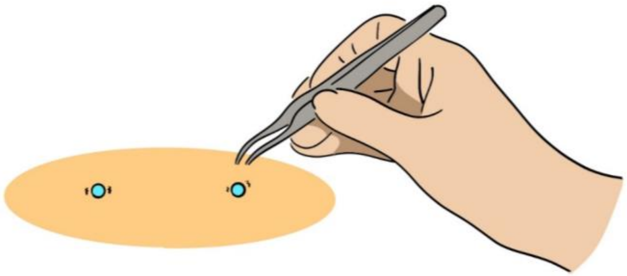
● : quarkonium

$\eta_c(\bar{c}c), \eta_b(\bar{b}b)$



Why finite temperature now?

~Motivation~



- ✓ Improve the control of **impurities**

  [S. Trotzky et al, . Nature Physics **8**, 325-330(2012)]

- ✓ Recent developments **in a finite temperature**

  The value of the potential takes on **a complex value**.

  [M. Laine et al. , J. High Energy Phys. 2007 (03), 054.]

  The real part of the potential at long distance between Heavy Polarons in **Superfluids**.

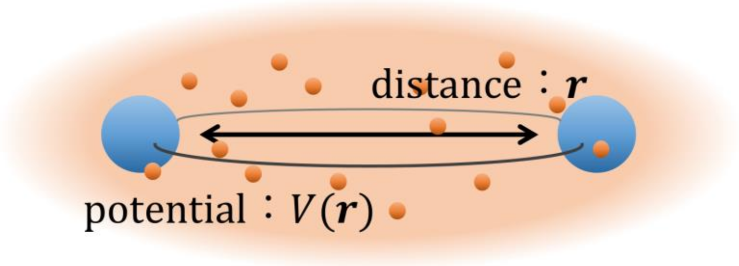
  $\propto \boldsymbol{r}^{-6}$ [Keisuke Fujii et al. , Phys. Rev. Lett. **129**, 233401(2024)]

  The imaginary part of the potential at long distance in **a cold atom system**.

  $\propto \boldsymbol{r}^{-2}$ [Y. Akamatsu et al. , Phys. Rev. A **110**, 033304 (2024)]

What behavior does the potential show

between quarkonia as impurities in a thermal pion gas?



：pion

：quarkonium

$\eta_c(\bar{c}c), \eta_b(\bar{b}b)$

~Theory~

✓ The Hamiltonian operator as a toy model :

$$\hat{H}_{\mathrm{T}} = \hat{H}_{\phi} + \hat{H}_{\pi} + \frac{g}{2}\int d^3\boldsymbol{x}\ \hat{\phi}^{\dagger}(\boldsymbol{x})\hat{\phi}(\boldsymbol{x})\left[\boldsymbol{\nabla}\hat{\pi}^{a}(\boldsymbol{x})\right]^2$$

✓ Definition of the potential : $\bar{V}(\boldsymbol{r}, \boldsymbol{\nabla}_r)$

→ Suppose **the correlation function** involving two quarkonia can be well described by **the two-body Schrödinger equation**.

- The correlation function :

$$\psi(\boldsymbol{r}, t) \equiv \int d^3\boldsymbol{R}\left\langle \hat{U}^{\dagger}(t,0)\hat{\phi}(\boldsymbol{R}-\boldsymbol{r}/2)\hat{\phi}(\boldsymbol{R}+\boldsymbol{r}/2)\hat{U}(t,0)\hat{\phi}^{\dagger}(\boldsymbol{x}_1)\hat{\phi}^{\dagger}(\boldsymbol{x}_2)\right\rangle$$

where

$$\langle\cdots\rangle = \mathrm{tr}\left[\cdots\hat{\rho}_{\mathrm{T}}\right] \qquad \hat{\rho}_{\mathrm{T}} = |0_{\phi}\rangle\langle 0_{\phi}| \otimes \hat{\rho}_{\pi}^{\mathrm{eq}}$$

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・ The two-body Schrödinger equation :

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{r}, t) \simeq \left[\cancel{-\frac{\boldsymbol{\nabla}_r^2}{2\mu}} + \bar{V}(\boldsymbol{r}, \cancel{\boldsymbol{\nabla}_r})\right]\psi(\boldsymbol{r}, t) \longrightarrow \psi(\boldsymbol{r}, t) \simeq e^{-i\bar{V}(\boldsymbol{r})t}\psi(\boldsymbol{r}, 0)$$

heavy mass

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where

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$$\boxed{\bar{V}(\boldsymbol{r}) = -\frac{1}{it}\boxed{?}}$$

compare

- The two-body Schrödinger equation :

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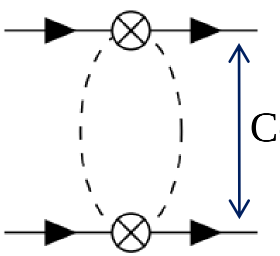
$$\bar{V}(\boldsymbol{r}) = \frac{2g}{t}\int_0^t dt' \left\langle \frac{1}{2}\left[\boldsymbol{\nabla}\hat{\pi}^{a}(\boldsymbol{0},t')\right]^2 \right\rangle + \frac{g^2}{it}\int_0^t dt' \int_0^t dt'' \left\{ \left\langle \mathrm{T}\; \frac{1}{2}\left[\boldsymbol{\nabla}\hat{\pi}^{a}(\boldsymbol{0},t')\right]^2 \frac{1}{2}\left[\boldsymbol{\nabla}\hat{\pi}^{b}(\boldsymbol{0},t'')\right]^2 \right\rangle_{\langle\pi^2(t)\rangle=0} + \underline{\left\langle \mathrm{T}\; \frac{1}{2}\left[\boldsymbol{\nabla}\hat{\pi}^{a}(\boldsymbol{r},t')\right]^2 \frac{1}{2}\left[\boldsymbol{\nabla}\hat{\pi}^{b}(\boldsymbol{0},t'')\right]^2 \right\rangle}_{\langle\pi^2(t)\rangle=0} \right\}$$

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) \simeq \left[\cancel{-\frac{\boldsymbol{\nabla}_r^2}{2\mu}} + \bar{V}(\boldsymbol{r},\cancel{\boldsymbol{\nabla}_r})\right]\psi(\boldsymbol{r},t) \longrightarrow \psi(\boldsymbol{r},t) \simeq e^{-i\bar{V}(\boldsymbol{r})t}\psi(\boldsymbol{r},0)$$

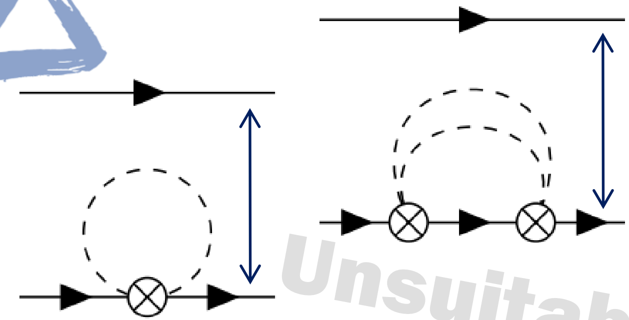
heavy mass

~Theory~

✓ The potential between quarkonia



Connected
interactions.



→ **Suitable as a potential** between quarkonia

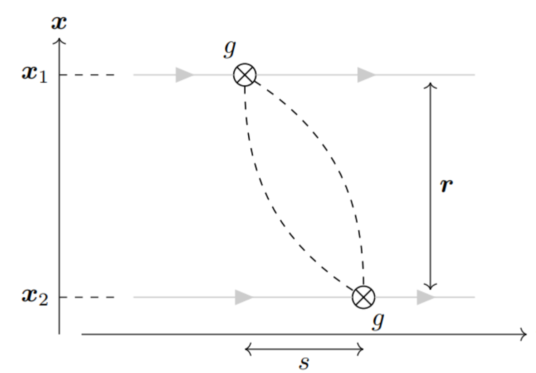
$$V(\boldsymbol{r}) = \frac{g^2}{it}\int_0^t dt' \int_0^t dt'' \left\langle \mathrm{T}\, \frac{1}{2}\left[\boldsymbol{\nabla}\hat{\pi}^a(\boldsymbol{r},t')\right]^2 \frac{1}{2}\left[\boldsymbol{\nabla}\hat{\pi}^b(\boldsymbol{0},t'')\right]^2 \right\rangle_{\langle\pi^2(t)\rangle=0}$$

✓ The imaginary part of the potential $V(\boldsymbol{r})$:

$$V_{\mathrm{Im}}(\boldsymbol{r}) = -\lim_{\omega\to 0}\frac{2T}{\omega}\,\mathrm{Im}\, G_{\mathrm{R}}(\boldsymbol{r},\omega)$$

where

$$G_{\mathrm{R}}(\boldsymbol{r},s) \equiv i\theta(s)\left\langle\left[\frac{g}{2}\left[\boldsymbol{\nabla}\hat{\pi}^a(\boldsymbol{r},s)\right]^2,\ \frac{g}{2}\left[\boldsymbol{\nabla}\hat{\pi}^b(\boldsymbol{0},0)\right]^2\right]\right\rangle_{\langle\pi^2(t)\rangle=0}$$



Leading-order potential is
the **Two-Pion Exchange Potential**

~Analysis~

✓ **The chiral lagrangian：**

$$\mathcal{L}_{\mathrm{T}} = \mathcal{L}_{\phi} + \mathcal{L}_{\pi} + \mathcal{L}_{\phi-\pi} + o\left(p^4\right)$$

$$\mathcal{L}_{\phi} = \phi^{\dagger}\left(i\partial_0 + \frac{\nabla^2}{2m_{\phi}}\right)\phi$$

$$\mathcal{L}_{\pi} = \frac{F^2}{4}\left\{\mathrm{tr}\left[\partial_{\mu}U\partial^{\mu}U^{\dagger}\right] + \mathrm{tr}\left[\chi^{\dagger}U + U^{\dagger}\chi\right]\right\}$$

$$\mathcal{L}_{\phi-\pi} = \phi^{\dagger}\phi\frac{F^2}{4}\left\{c_0\,\mathrm{tr}\left[\partial_0 U\partial^0 U^{\dagger}\right] + c_i\,\mathrm{tr}\left[\partial_i U\partial^i U^{\dagger}\right] + c_m\,\mathrm{tr}\left[\chi^{\dagger}U + U^{\dagger}\chi\right]\right\}$$

- $\mathcal{L}_{\mathrm{T}}$ has the symmetry $\mathbf{SU(2)}_{L} \otimes \mathbf{SU(2)}_{R}$
- NG modes：$U(x) = e^{i\Phi(x)/F}$ where $\Phi(x) = \tau^a\pi^a(x)$, τ^a is the Pauli matrices.
- $\chi = \begin{pmatrix} m_{\pi}^2 & 0 \\ 0 & m_{\pi}^2 \end{pmatrix}$
- c_0, c_i, c_m are coupling constants

$$\Rightarrow \mathcal{L}_{\mathrm{T}} = \phi^{\dagger}\left(i\partial_0 + \frac{\nabla^2}{2m_{\phi}}\right)\phi + \frac{1}{2}\pi^a\left(-\partial_{\mu}\partial^{\mu} - m_{\pi}^2\right)\pi^a + \cdots$$

$$+ \phi^{\dagger}\phi\left\{\frac{1}{2}\left[c_0\left(\partial_0\pi^a\right)^2 - c_i\left(\nabla\pi^a\right)^2 - c_m m_{\pi}^2\left(\pi^a\right)^2\right] + \cdots\right\}$$

~Analysis~

✓ Introduce **the Hamiltonian** through a Legendre transformation：

$$\mathcal{H}_{\mathrm{T}} = \Phi^{\dagger}\partial_0\phi^{\dagger} + \Phi\partial_0\phi + \Pi^a\partial_0\pi^a - \mathcal{L}_{\mathrm{T}} = \mathcal{H}_{\phi} + \mathcal{H}_{\pi} + \mathcal{H}_{\phi-\pi} + \mathcal{H}_{\pi-\pi}$$

$$\mathcal{H}_{\phi} = \phi^{\dagger}\left(-\frac{\nabla^2}{2m_{\phi}}\right)\phi$$

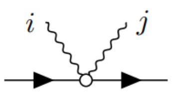
$$\mathcal{H}_{\pi} = \frac{1}{2}\left(\Pi^a\right)^2 + \frac{1}{2}\left(\nabla\pi^a\right)^2 + \frac{1}{2}m_{\pi}^2\left(\pi^a\right)^2$$

$$\mathcal{H}_{\phi-\pi} = \frac{1}{2}\left\{\left[1 + c_0\phi^{\dagger}\phi(1+\cdots)\right]^{-1} - 1\right\}(\Pi^a)^2 + \phi^{\dagger}\phi\left\{\frac{1}{2}\left[c_i\left(\nabla\pi^a\right)^2 + c_m m_{\pi}^2\left(\pi^a\right)^2\right] + \cdots\right\}$$

✓ **Feynman rule**

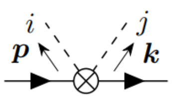
$$\widetilde{\Delta}_{\mathrm{M}}^{\pi-\pi}(\omega_n,\boldsymbol{p}) = \xrightarrow{\omega_n,\boldsymbol{p}} = \frac{1}{\omega_n^2+E_p^2}\delta^{ij}$$

$$\widetilde{\Delta}_{\mathrm{M}}^{\Pi-\Pi}(\omega_n,\boldsymbol{p}) = \xrightarrow{\omega_n,\boldsymbol{p}} = \frac{E_p^2}{\omega_n^2+E_p^2}\delta^{ij}$$



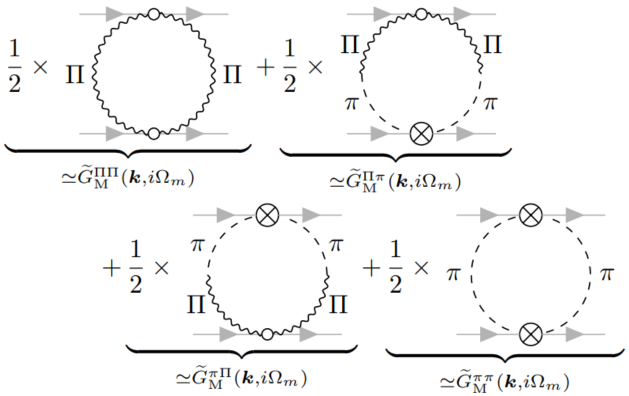
$$= -\delta^{ij}c_0$$

$$\widetilde{\Delta}_{\mathrm{M}}^{\pi-\Pi}(\omega_n,\boldsymbol{p}) = \xrightarrow{\omega_n,\boldsymbol{p}} = \frac{\omega_n}{\omega_n^2+E_p^2}\delta^{ij}$$



$$= -\delta^{ij}\left(c_i\boldsymbol{p}\cdot\boldsymbol{k} - c_m m_{\pi}^2\right)$$

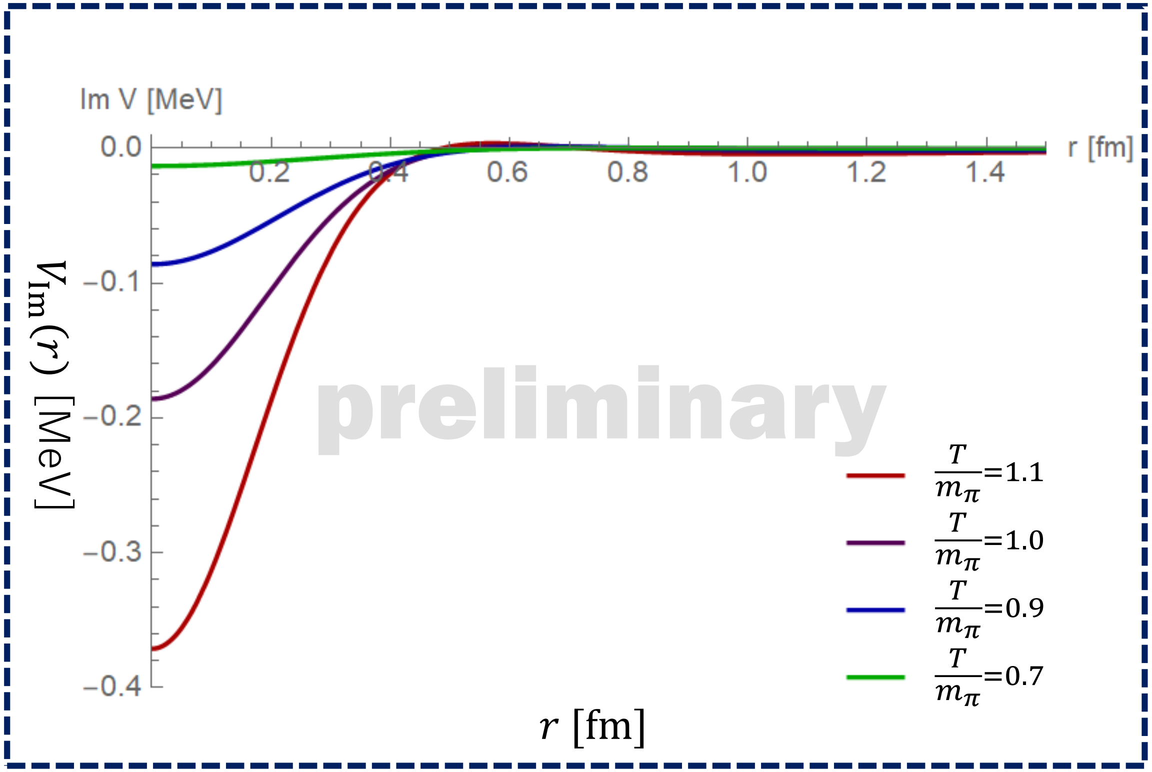
$$\widetilde{\Delta}_{\mathrm{M}}^{\Pi-\pi}(\omega_n,\boldsymbol{p}) = \xrightarrow{\omega_n,\boldsymbol{p}} = \frac{-\omega_n}{\omega_n^2+E_p^2}\delta^{ij}$$



~Result~

✓ **The imaginary part** of the potential

$$V_{\mathrm{Im}}(r) = -3\pi \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{r}} \left\{ c_0 E_k^2 + \left[c_i \left(\boldsymbol{k}\cdot\boldsymbol{q} \right) - c_m m_\pi^2 \right] \right\}^2 \frac{1}{E_k^2} n_B(E_k) \left[1 + n_B(E_k) \right] \delta(E_k - E_q)$$



where

$$E_k^2 = \boldsymbol{k}^2 + m_\pi^2$$

$$n_B(E_k) = \frac{1}{e^{\beta E_k} - 1}$$

$$c_0 = -2.55 \text{ GeV}^{-3}$$

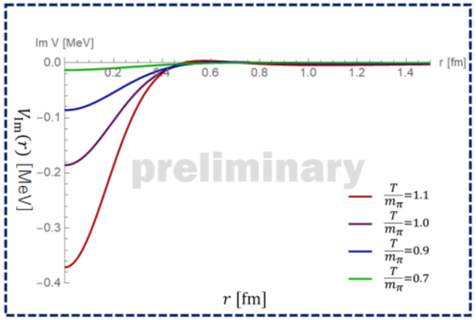
$$c_i = -5.00 \text{ GeV}^{-3}$$

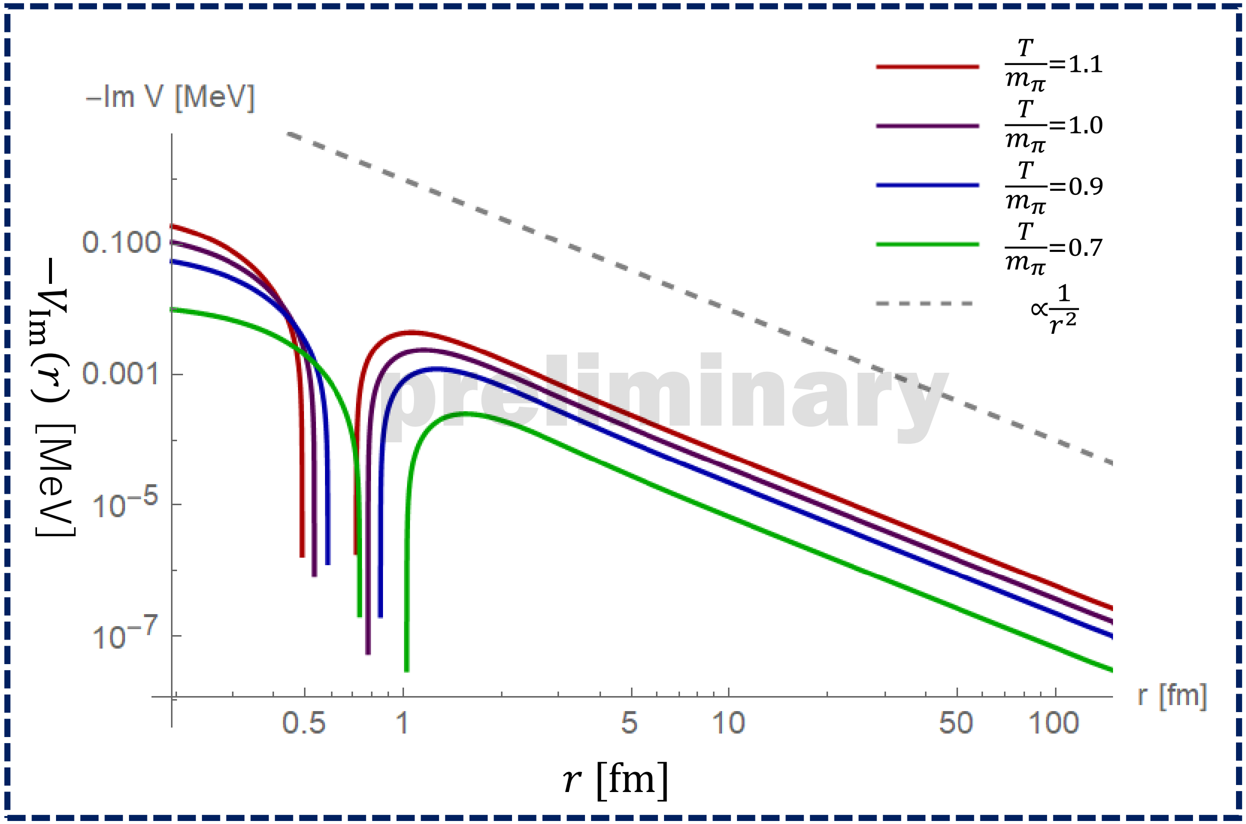
$$c_m = -6.58 \text{ GeV}^{-3}$$

~Result~

✓ Behavior the imaginary part of the potential **at long distance**

× (−1) and log-log plot





$\boldsymbol{r^{-2}}$ behavior at long distance

~Conclusion~

~Summary~

- ✓ The potential between heavy quarkonia in a thermal pion gas has an imaginary part due to interactions with thermal pions.
- ✓ The imaginary part of the potential is found to show r^{-2} behavior at long distance.

~ Future work ~

- ✓ Analysis of the real part of the potential.
- ✓ Applications to exotic states of hadrons such as mesonic molecule states.
- ✓ Analysis of other models like those including nucleons.

Thank you for your attention.