Phase diagram of QCD matter with magnetic field: domain-wall Skyrmion chain in chiral soliton lattice

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Symmetry and Effective Field Theory of Quantum Matter
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JHEP 12 (2023) 032

Outline

- Introduction

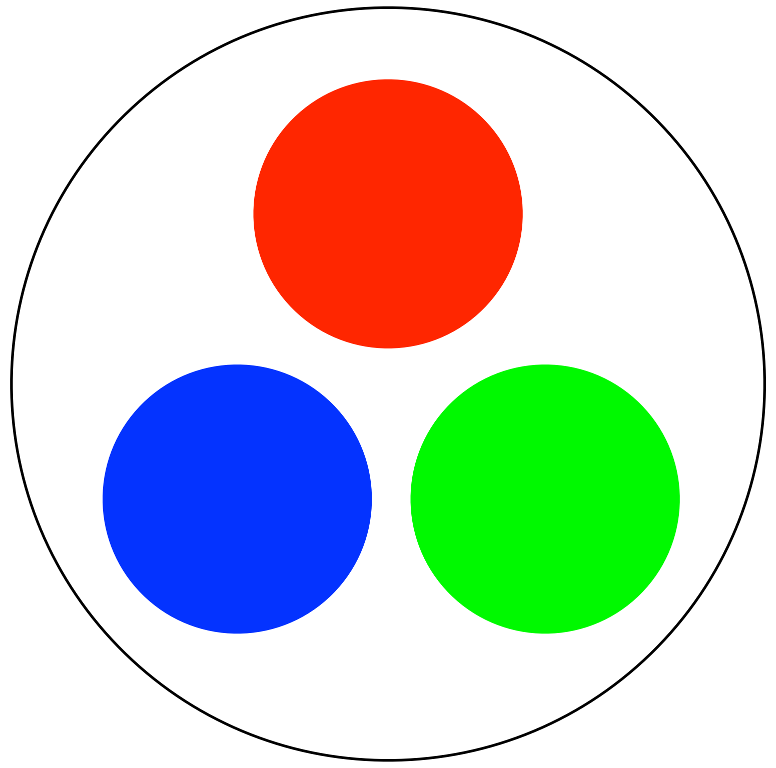
- Chiral soliton lattice Son and Stephanov (2008); Brauner and Yamamoto (2017)

- Domain wall Skyrmion Eto, KN and Nitta, JHEP 12 (2023) 032

Baryon

- Elementary Degrees of freedom = Quarks (up, down)

- Baryon = Particle composed of quarks



$$\simeq \quad \epsilon^{a_1 a_2 a_3} q^{f_1}_{a_1} q^{f_2}_{a_2} q^{f_3}_{a_3}$$

Meson

- **Nambu-Goldstone (NG) mode of spontaneous chiral symmetry breaking**

$$\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{L}} \to \mathrm{SU}(2)_{\mathrm{V}}$$

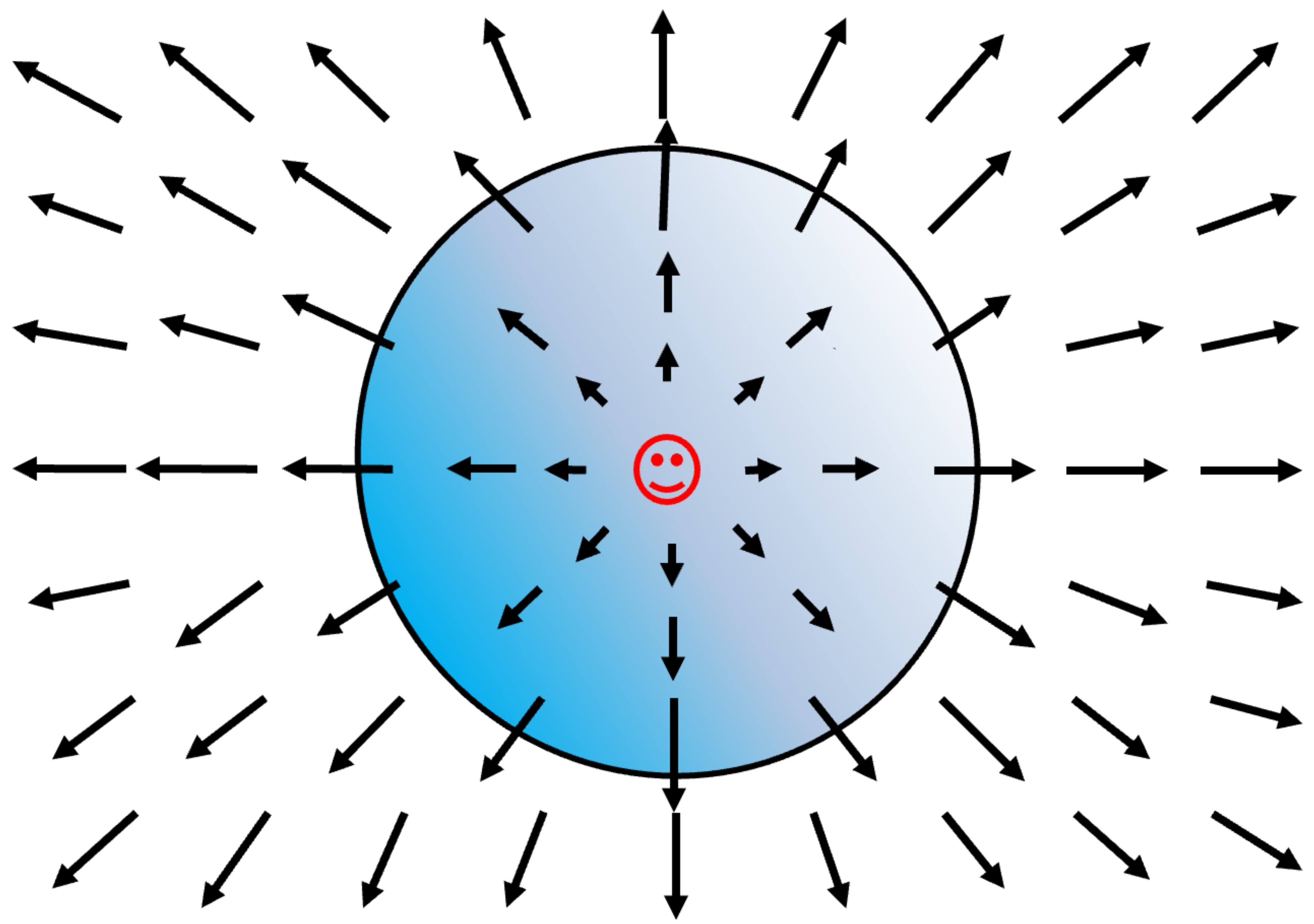
- **Degrees of freedom of the low-energy dynamics = Pions**

$$\Sigma(x) = \exp\left(\frac{\mathrm{i}\pi_a \tau_a}{f_\pi}\right)$$

Skyrmions

- Can the baryons be made by pions (rather than quarks)?

Baryon as soliton = Skyrmion



Topological number = Baryons

$$N_{\mathrm{B}} = \frac{1}{24\pi^2}\int \mathrm{d}^3x\, \epsilon^{ijk}\mathrm{tr}(\Sigma\partial_i\Sigma^\dagger\Sigma\partial_j\Sigma^\dagger\Sigma\partial_k\Sigma^\dagger)$$

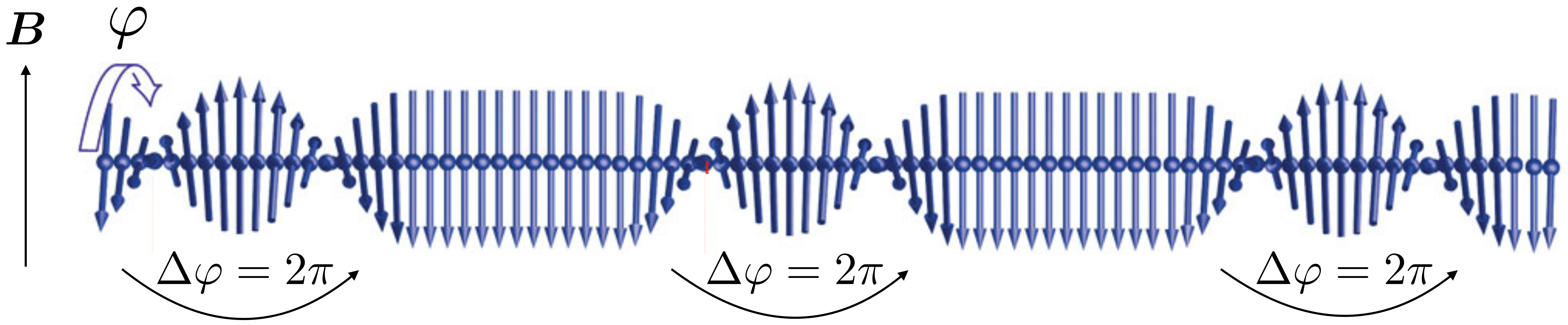
- How many times R^3 surrounds the configuration space of the pions S^3.

Solitonic phase

- Generally, topological solitons are energetically unstable…

- You can find the case where solitons appear in ground states!

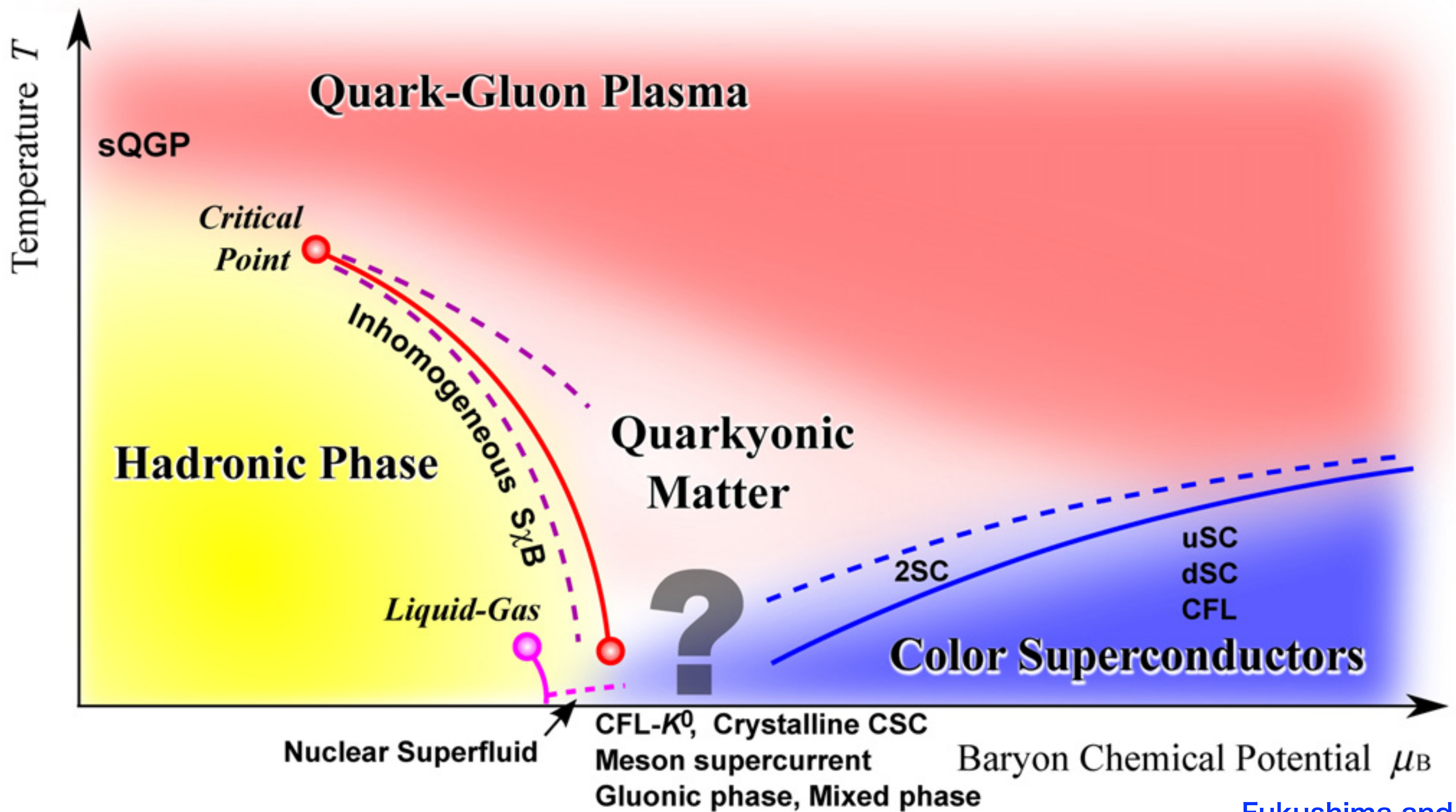
Chiral magnet



The twisting spin structure is spatially localized = soliton

- Other examples are Baby Skyrmion crystals and Abrikosov lattice …

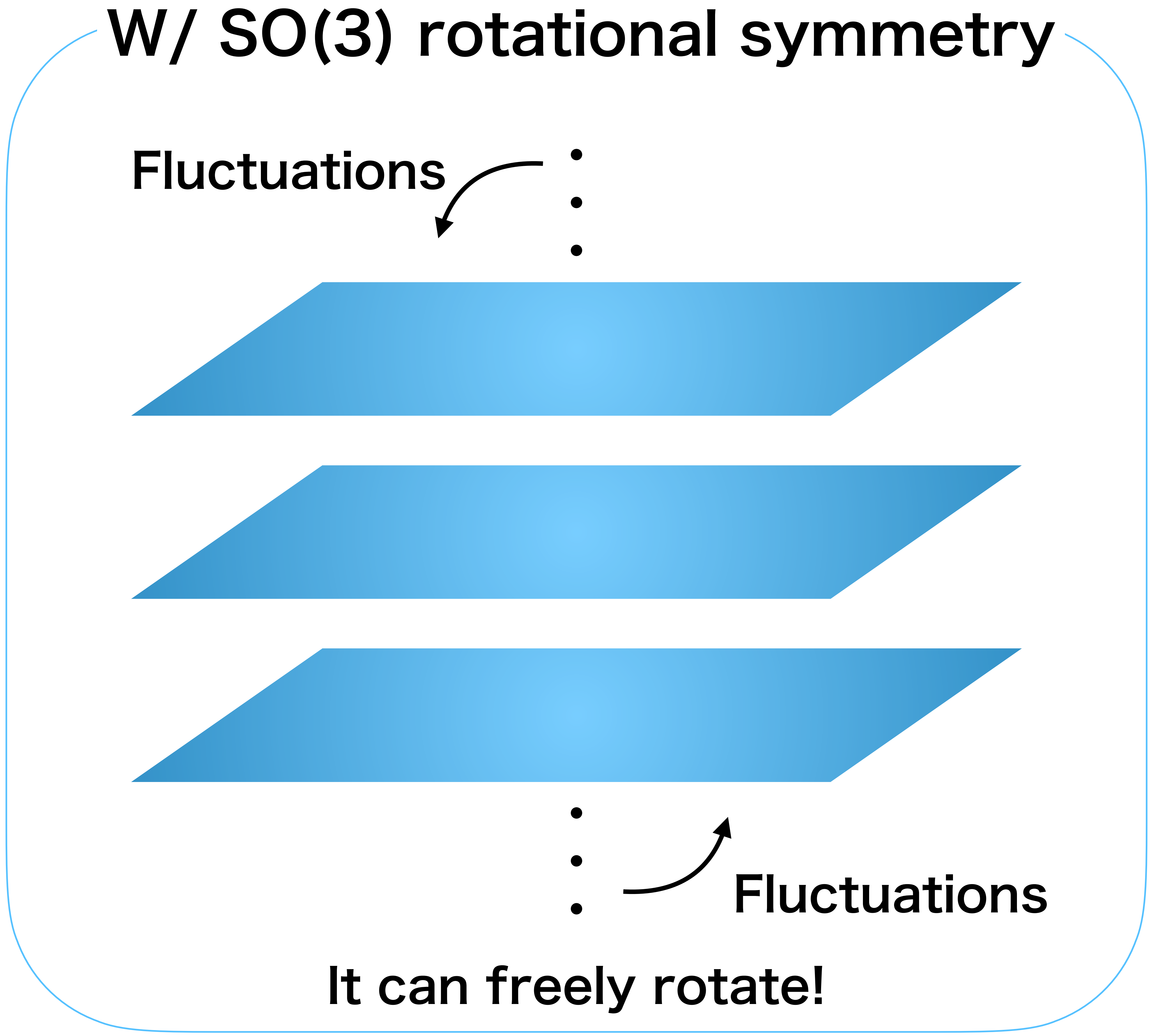
QCD phase diagram



Fukushima and Hatsuda (2008)

Instability of inhomogeneous state

- Please imagine a field $\phi(z)$ depends on z and exhibits periodic behavior.



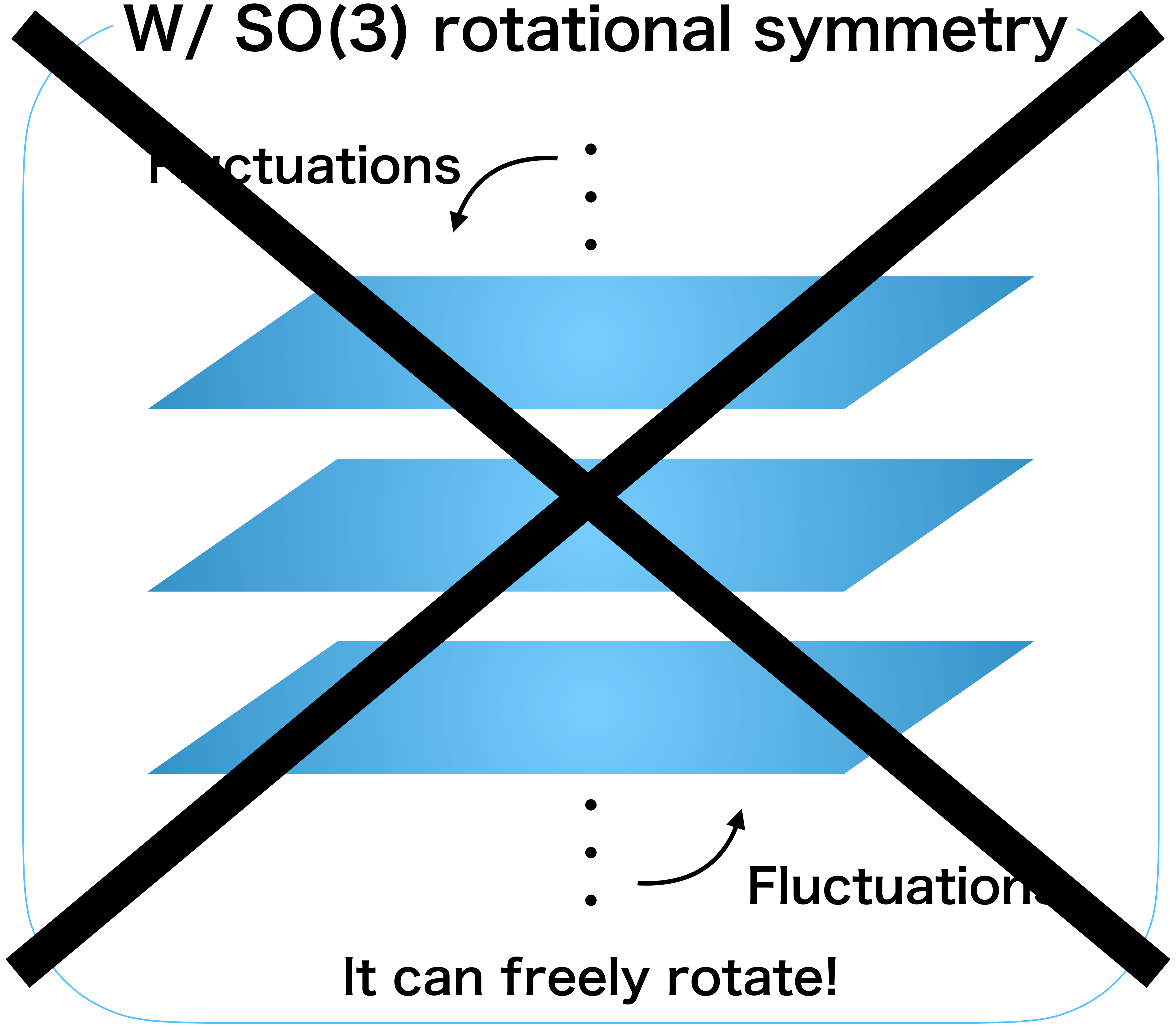
Hidaka, Kamikado, Kanazawa and Noumi (2015)

Lee, Nakano, Tsue, Tatsumi and Friman (2015)

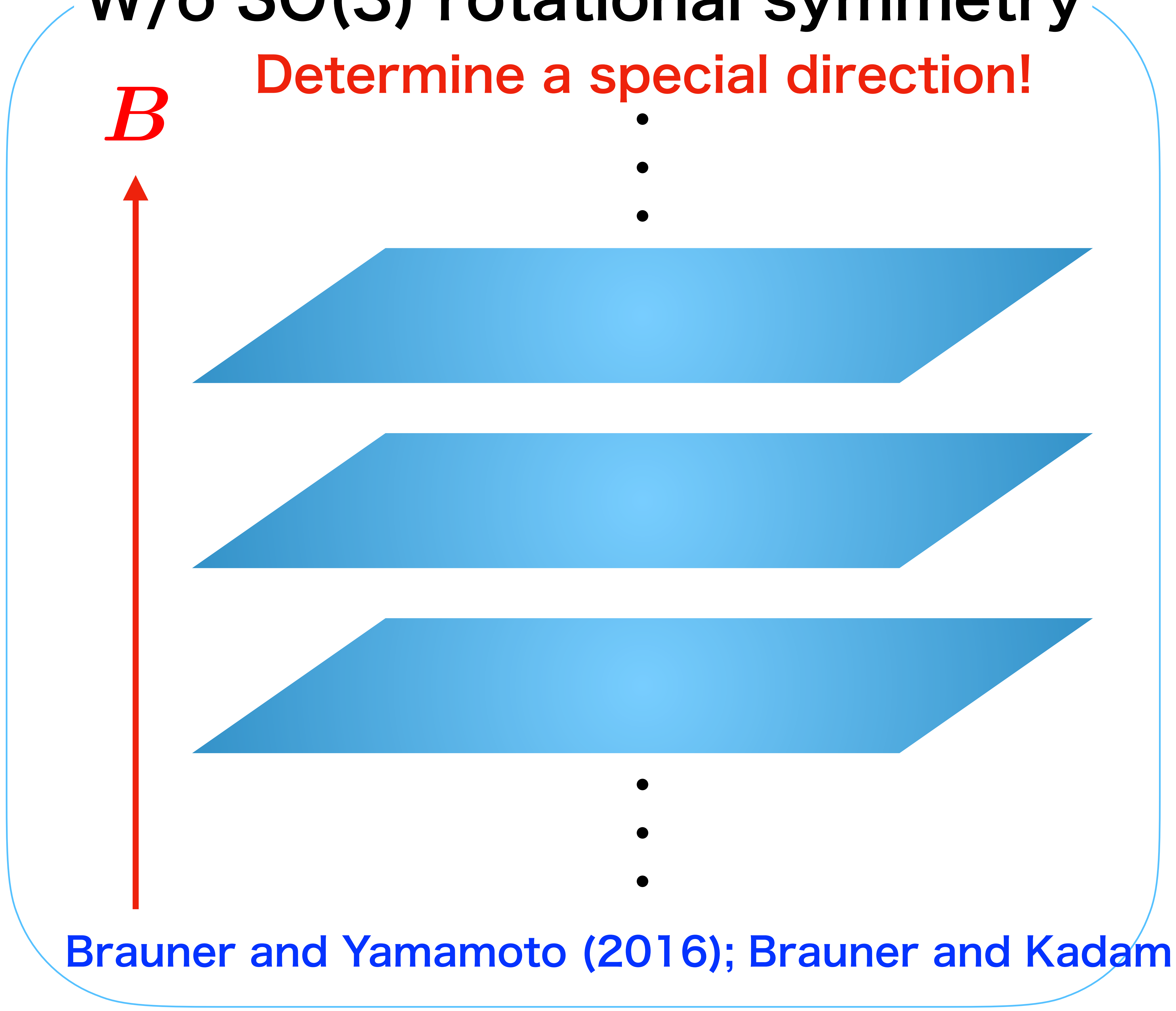
Instability of inhomogeneous state

- Please imagine a field $\phi(z)$ depends on z and exhibits periodic behavior.

W/ SO(3) rotational symmetry

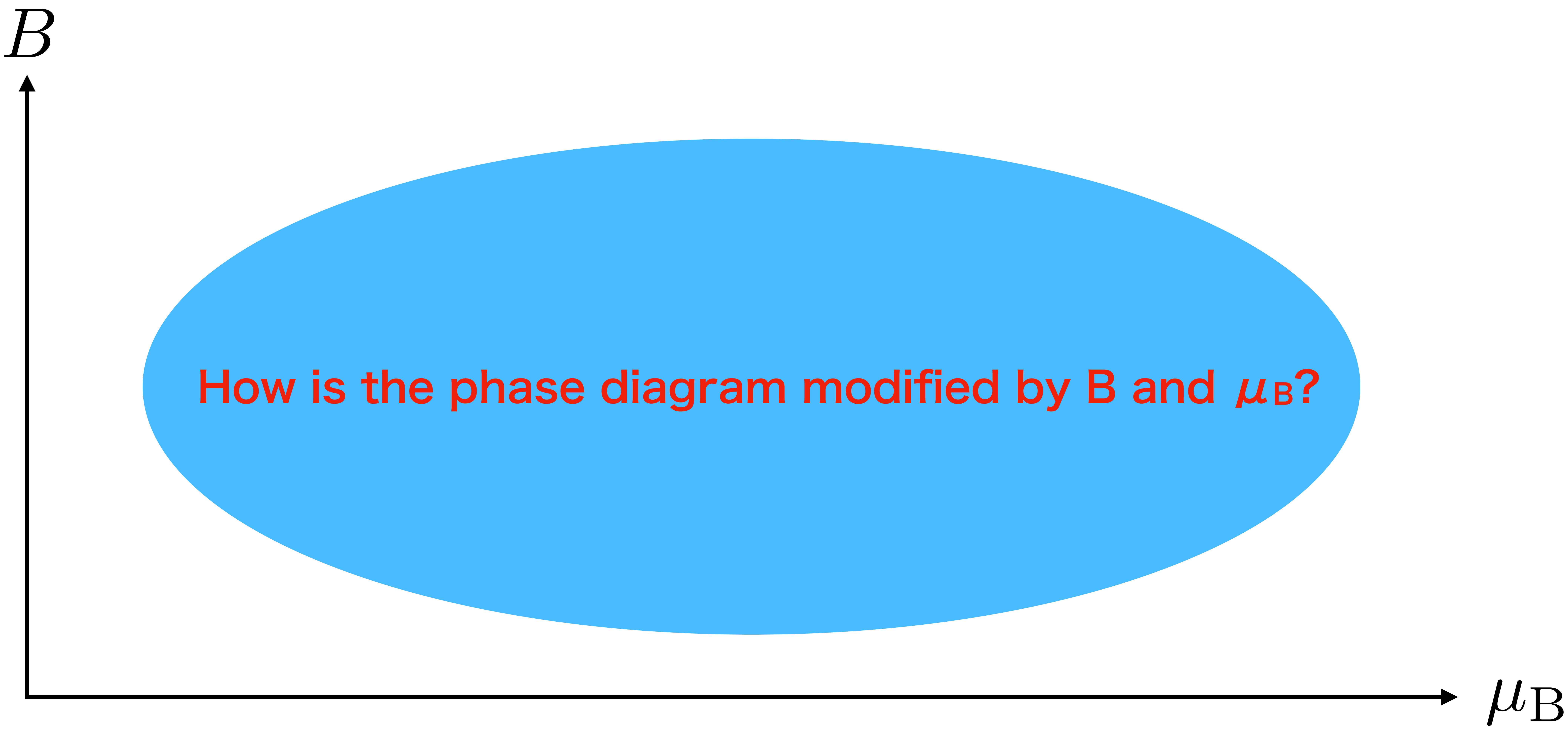


W/o SO(3) rotational symmetry



Brauner and Yamamoto (2016); Brauner and Kadam (2017)

What I want to discuss today



What I want to discuss today

- Consider zero-temperature case.

- I will use the chiral perturbation theory.
  - Consider the finite-B modification in a region with a relatively small μ_B.

Since pions do not carry baryon number, nothing seems to happen even if μ_B is considered.

- Skyrmion plays an important role to determine the phase structure.

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Chiral perturbation theory

- **Order parameter is the chiral condensate:** $\langle \bar{q}q \rangle = |\langle \bar{q}q \rangle| \Sigma$

- **Nambu-Goldstone boson:** $\Sigma = \exp(\mathrm{i}\sigma_a \phi_a)\,, \quad \phi_a \equiv \pi_a / f_\pi$

- **Effective Lagrangian:** $$\mathcal{L}_{\mathrm{ChPT}} = \frac{f_\pi^2}{4} \mathrm{tr}\left(D_\mu \Sigma D^\mu \Sigma^\dagger \right) - \frac{f_\pi^2 m_\pi^2}{4} (2 - \Sigma - \Sigma^\dagger)$$

$$D_\mu \Sigma = \partial_\mu \Sigma + \mathrm{i} A_\mu [Q, \Sigma]\,, \quad Q = \mathrm{diag}(2/3, -1/3)$$

ChPT w/ topological terms

- Baryon current couples to $U(1)_B$ gauge field (minimal coupling):

$$\mathcal{L}_{\mathrm{B}} = -A^{\mu}_{\mathrm{B}} j_{\mathrm{B}\mu} \, , \quad A^{\mu}_{\mathrm{B}} = (\mu_{\mathrm{B}}, \mathbf{0})$$

- "trial and error" $U(1)_{em}$ gauging while preserving baryon number conservation.

$$j^{\mu}_{\mathrm{B}} = \underbrace{-\frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}\{L_\nu L_\alpha L_\beta}_{\text{Skyrimon charge}} - \underbrace{3ie\partial_\nu\left[A_\alpha Q(L_\beta + R_\beta)\right]}_{U(1)_{em}\text{ gauged part}}\}$$

$$L_\mu \equiv \Sigma\partial_\mu\Sigma^\dagger \, , \; R_\mu \equiv \partial_\mu\Sigma^\dagger\Sigma$$

$$Q = \mathrm{diag}(2/3, -1/3)$$

Son and Stephanov (2008); Goldstone and Wilczek (1981); Witten (1983)

sine-Gordon theory with the topological term

- **I first ignore $\pi_\pm$:** $\Sigma = \mathrm{e}^{\mathrm{i}\phi_3 \tau_3}$

- **Reduced Hamiltonian (B is oriented in z-direction) :**

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z \phi_3)^2 + f_\pi^2 m_\pi^2 (1 - \cos\phi_3) - \frac{e\mu_\mathrm{B}}{4\pi^2} B \partial_z \phi_3$$

- The last term stems from the 2nd term of the skyrmion term.

$$\mathcal{L}_\mathrm{B} = -\mu_\mathrm{B} \frac{\epsilon^{0ijk}}{24\pi^2} \mathrm{tr}\{\cancel{L_i L_j L_k} - \underline{3\mathrm{i}e\partial_i \left[A_j Q(L_k + R_k)\right]}\}$$

- **B≠0 → Finite 1st derivative term → Favor ϕ inhomogenity**

- **What is a ground state at finite B?**

Brauner and Yamamoto (2017)

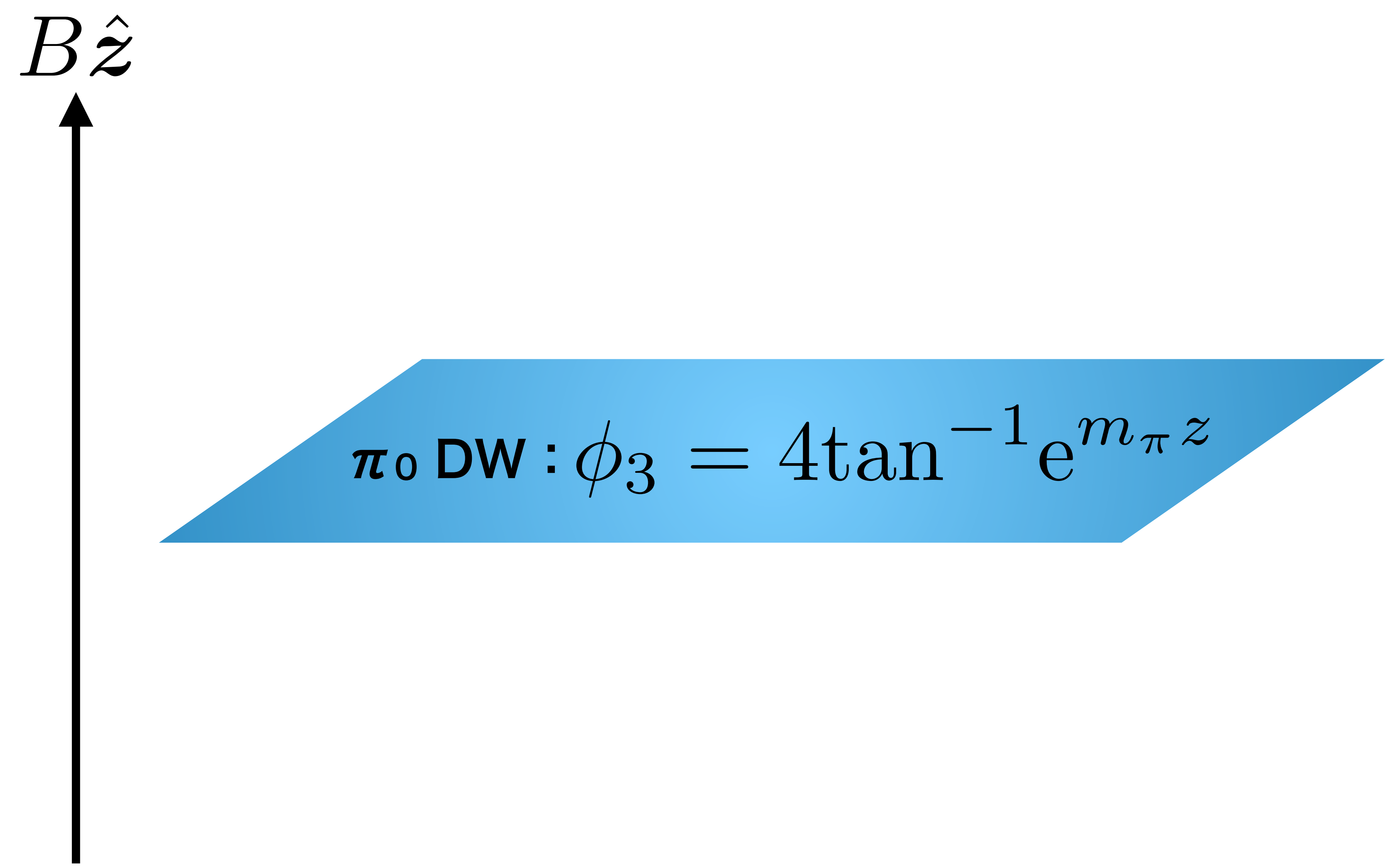
Chiral Soliton Lattice

- **EOM :** $\partial_z^2 \phi_3 = m_\pi^2 \sin\phi_3$

$$\phi_3(-\infty) = 0\,,\ \phi_3(\infty) = 2\pi$$

- **Energy :** $E = \int_{-\infty}^{\infty} \mathrm{d}z\, \mathcal{H} = 8 m_\pi^2 f_\pi - \dfrac{e\mu_{\mathrm{B}} B}{2\pi}$

- **Critical B :** $B_{\mathrm{CSL}} = \dfrac{16\pi m_\pi f_\pi^2}{e\mu_{\mathrm{B}}}$



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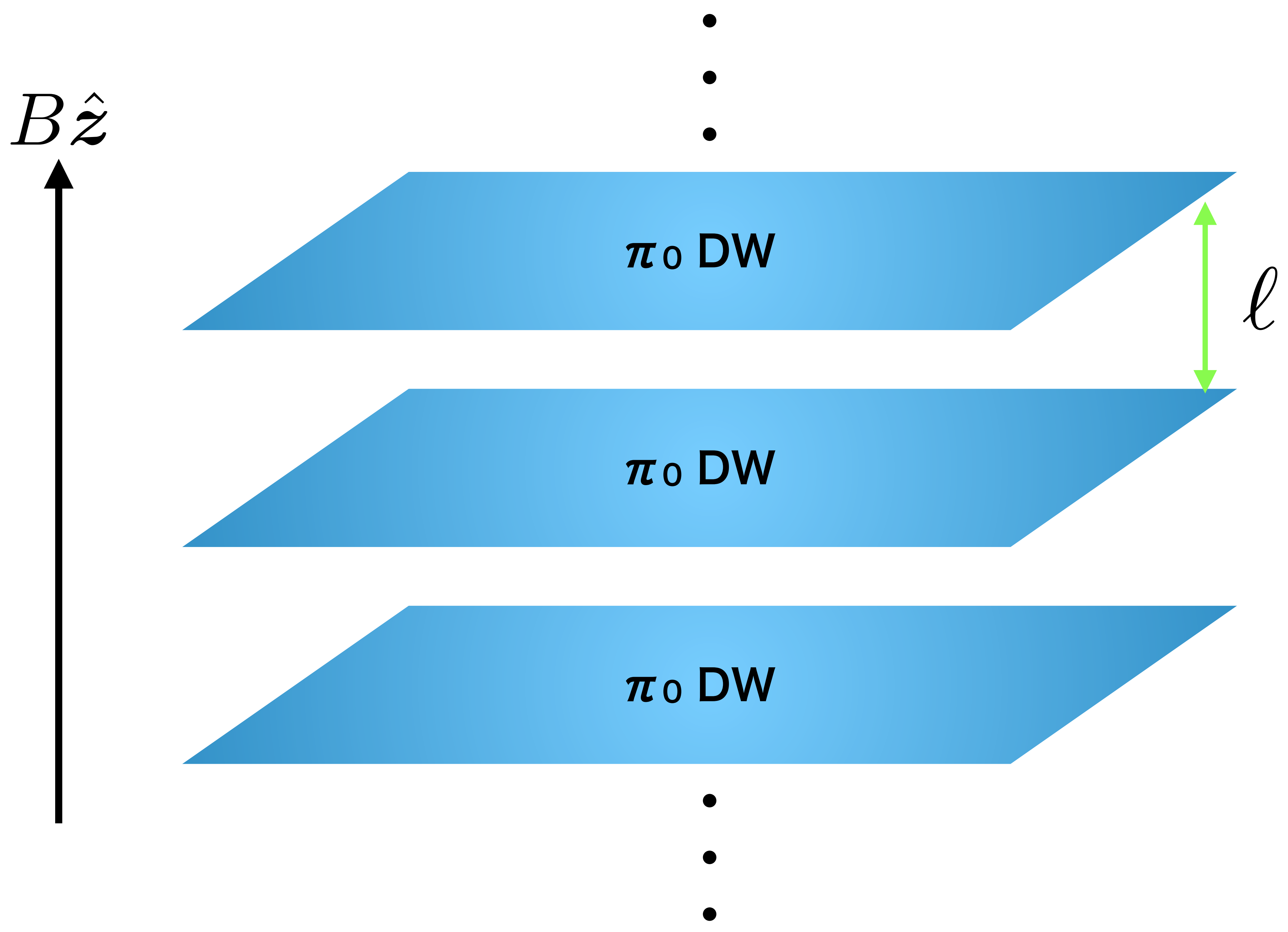


- **Pack many DWs in ground state!**
- **Impossible to pack due to the repulsive force.**

Dautry and Nyman (1979); Hatsuda (1986); Son and Stephanov (2008); Nishiyama, Karasawa and Tatsumi (2015); Brauner and Yamamoto (2017)

Chiral Soliton Lattice

- **EOM = Pendulum:** $\partial_z^2 \phi_3 = m_\pi^2 \sin \phi_3$

  $\phi_3(0) = \pi\,,\ \phi_3(\ell) = 2\pi$

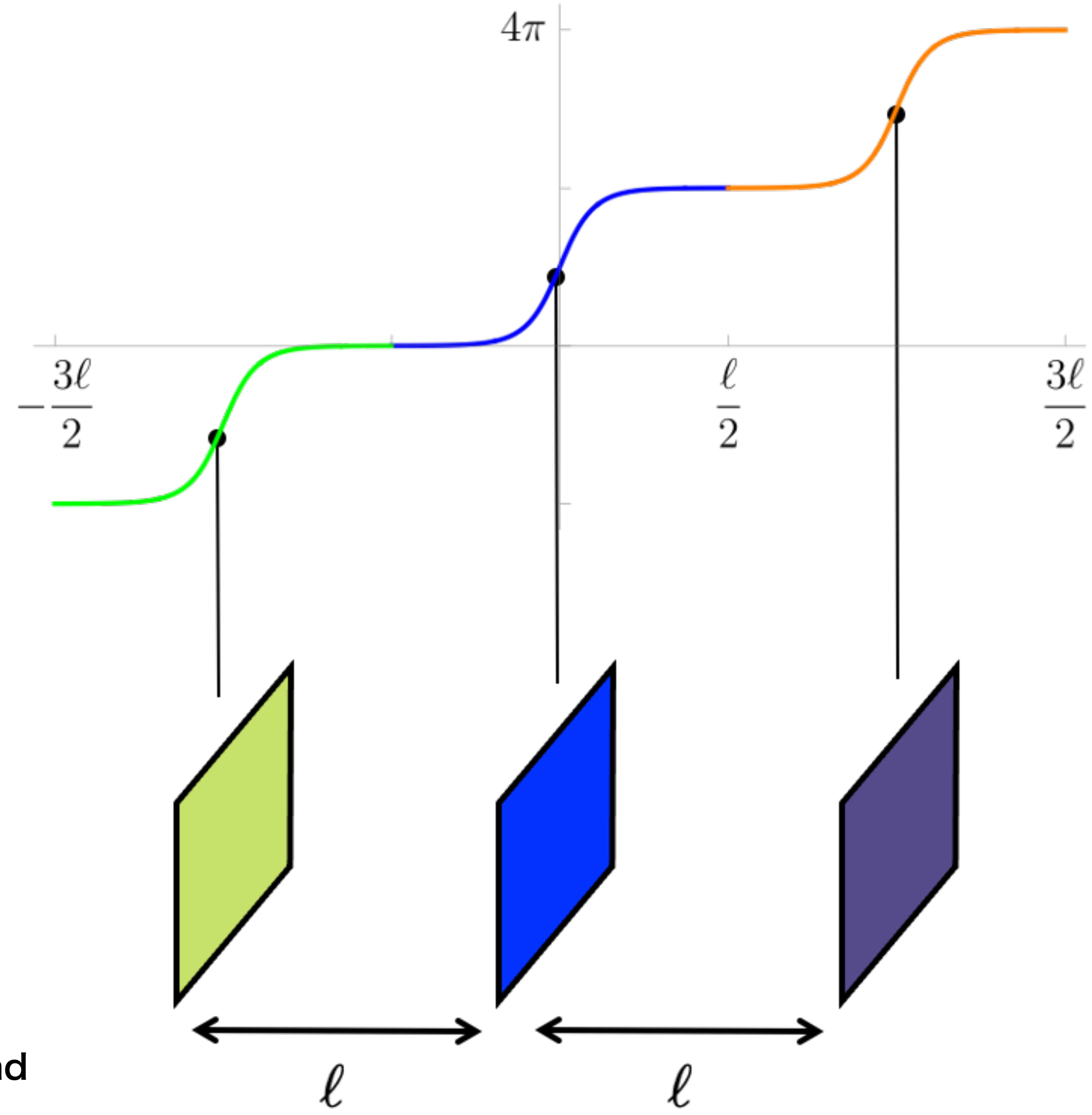
- **Analytic solution :** $\bar{\phi} = 2\mathrm{am}\left(z/\kappa, \kappa\right) + \pi$

  κ : Elliptic modulus

- **Period :** $\phi(z+\ell) = \phi(z) + 2\pi$

  $\ell = 2\kappa K(\kappa)$

  $K(\kappa)$: The complete elliptic integral of the first kind



Brauner and Yamamoto (2017)

Minimization of the total energy

- Minimizing the total energy gives us the optimal κ.

Brauner and Yamamoto (2017)

$$\mathcal{E}_{\rm tot} = \int_0^\ell \mathrm{d}z \left[\underbrace{\frac{f_\pi^2}{2}(\partial_z\phi)^2 + f_\pi^2 m_\pi^2 (1-\cos\phi)}_{\text{positive}} - \underbrace{\frac{\mu_{\rm B}}{4\pi^2} B \partial_z \phi}_{\text{negative!}} \right]$$

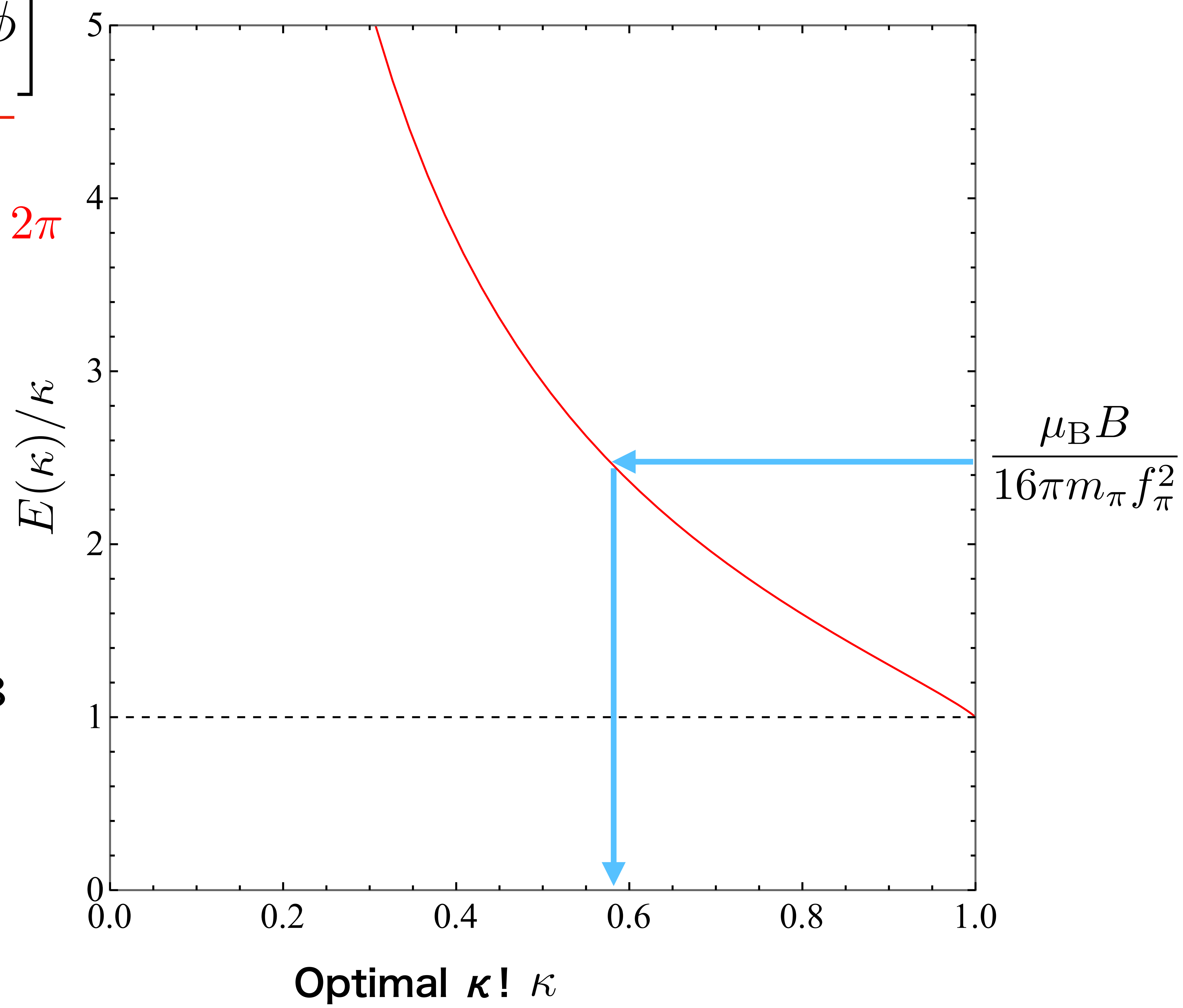
$\phi(\ell) - \phi(0) = 2\pi$

- Energy minimization condition :

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\mathcal{E}_{\rm tot}}{\ell}\right) \to \frac{E(\kappa)}{\kappa} = \frac{\mu_{\rm B} B}{16\pi m_\pi f_\pi^2}$$

$E(\kappa)$: The complete elliptic integral of the 2nd kind

- Critical magnetic field : $B_{\rm CSL} = 16\pi f_\pi^2 m_\pi / \mu_{\rm B}$



- The energy density with the minimization condition is smaller than that of ϕ_3=0.

Fluctuations of $\pi_{\pm}$

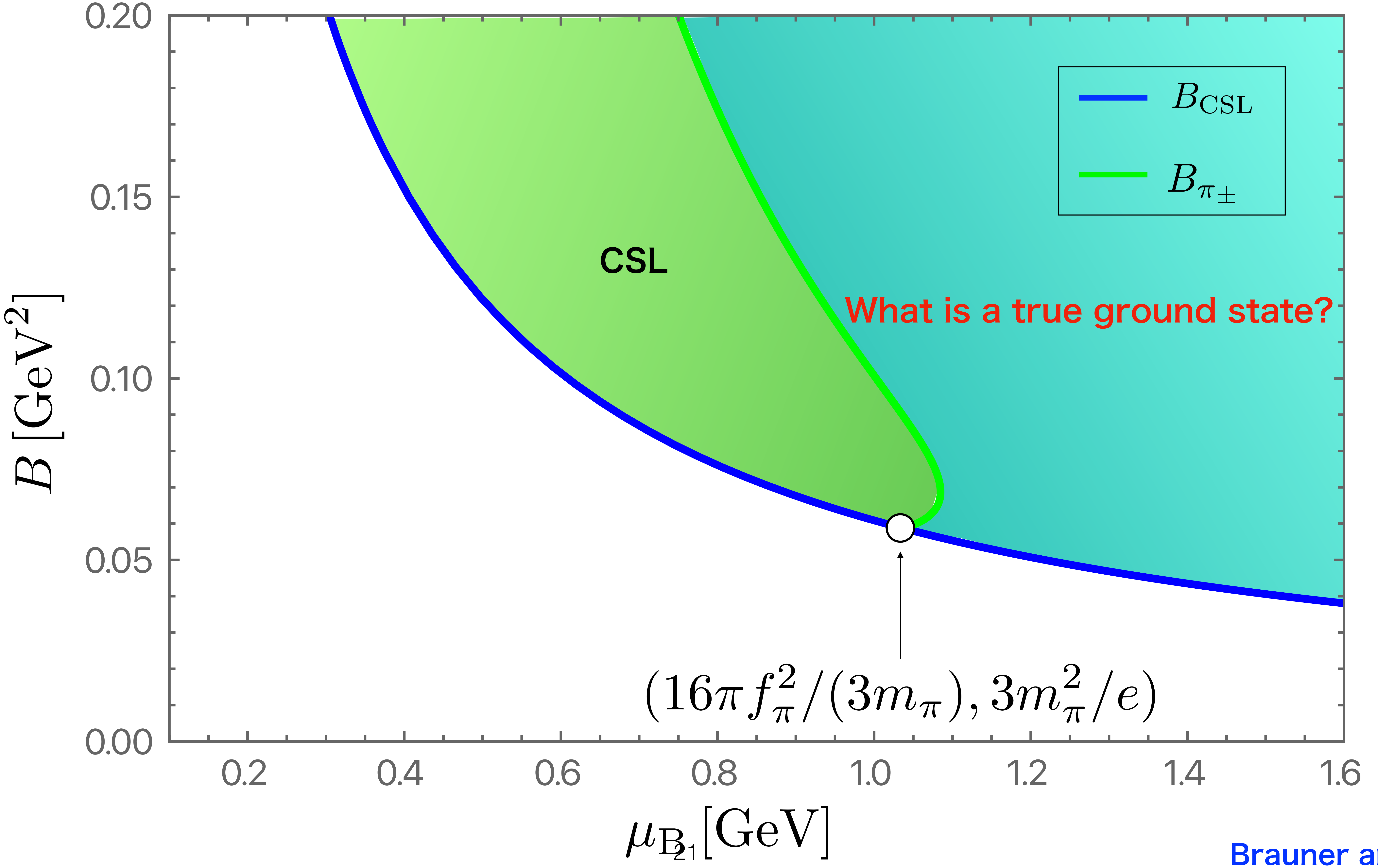
- **Fluctuation around the CSL background :**
- **CSL is unstable against fluctuations of $\pi_{\pm}$ above $B^{\pi\pm}_{BEC}$**

$$\boxed{\frac{E(k)}{k} = \frac{\mu_B B^{\pi\pm}_{\rm BEC}}{16\pi m_\pi f_\pi^2}} \qquad \boxed{B^{\pi\pm}_{\rm BEC} = \frac{m_\pi^2}{k^2}\left(2 - k^2 + 2\sqrt{1-k^2+k^4}\right)}$$

$$k = k(B^{\pi\pm}_{\rm BEC})$$

- **Derive the effective action up to the 2nd of the fluctuations from the CSL**
- **Calculate the dispersion relation ω**
- **When $\omega^2<0$, the fluctuation is tachyonic and CSL becomes unstable.** Brauner and Yamamoto (2017)

μ_B-B phase diagram



Brauner and Yamamoto (2017)

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What we overlooked

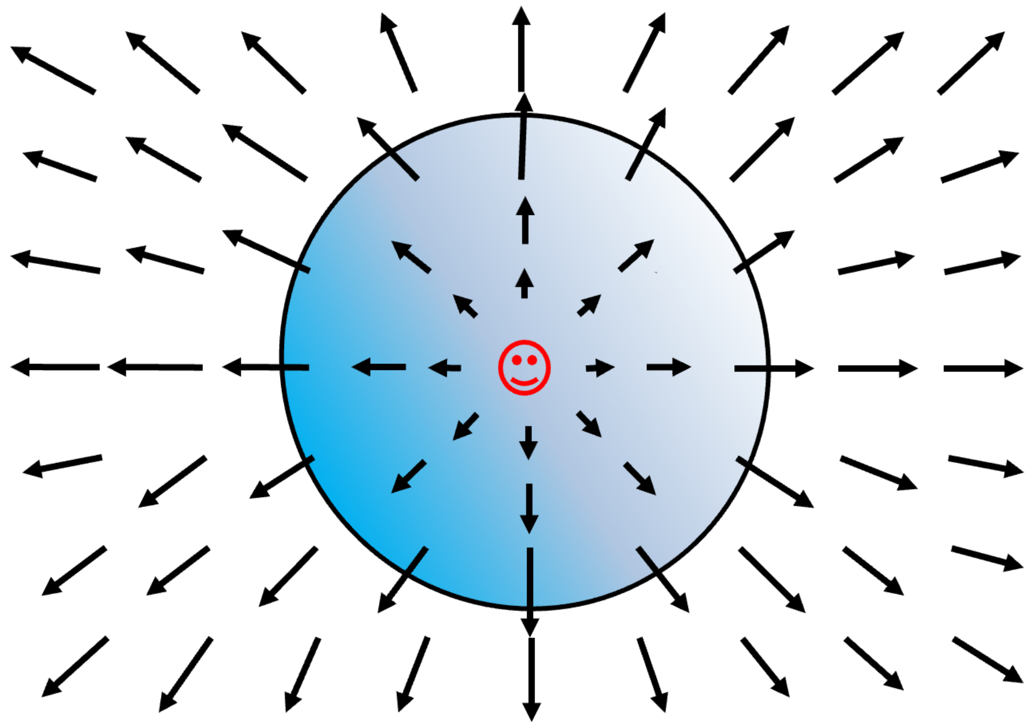
- The baryon current contains the Skyrmion charge, which is O(p²).

$$\mathcal{L}_{\mathrm{B}} = -\mu_{\mathrm{B}} \frac{\epsilon^{0ijk}}{24\pi^2} \mathrm{tr}\{\underline{L_i L_j L_k} - \underbrace{3ie\partial_i \left[A_j Q(L_k + R_k)\right]}_{\mu_{\mathrm{B}} B \partial_z \phi_3 \subset \mathcal{L}_{\mathrm{B}}}\}$$

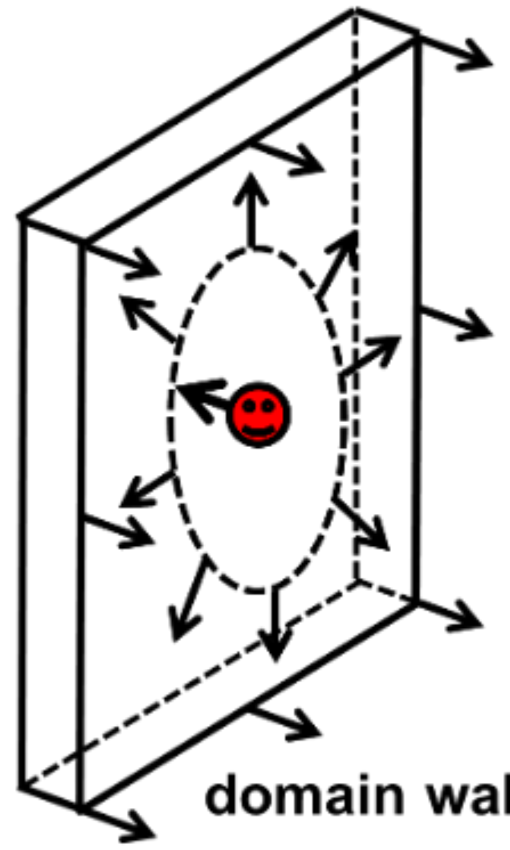
- In the unstable region, $\pi_\pm$ is important element.
- The Skyrmion charge term becomes finite only when $\pi_\pm$ is considered.

- When Σ has Skymion number, 1st term decreases the energy density!

Skyrmion : $\pi_3(S^3)$



Baby Skyrmion : $\pi_2(S^2)$



become stable at finite μ_{B}!

Non-Abelian soliton

- **The single soliton:**

$$\Sigma_0 = \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,,\ \ \theta = 4\tan^{-1}\mathrm{e}^{m_\pi z}$$

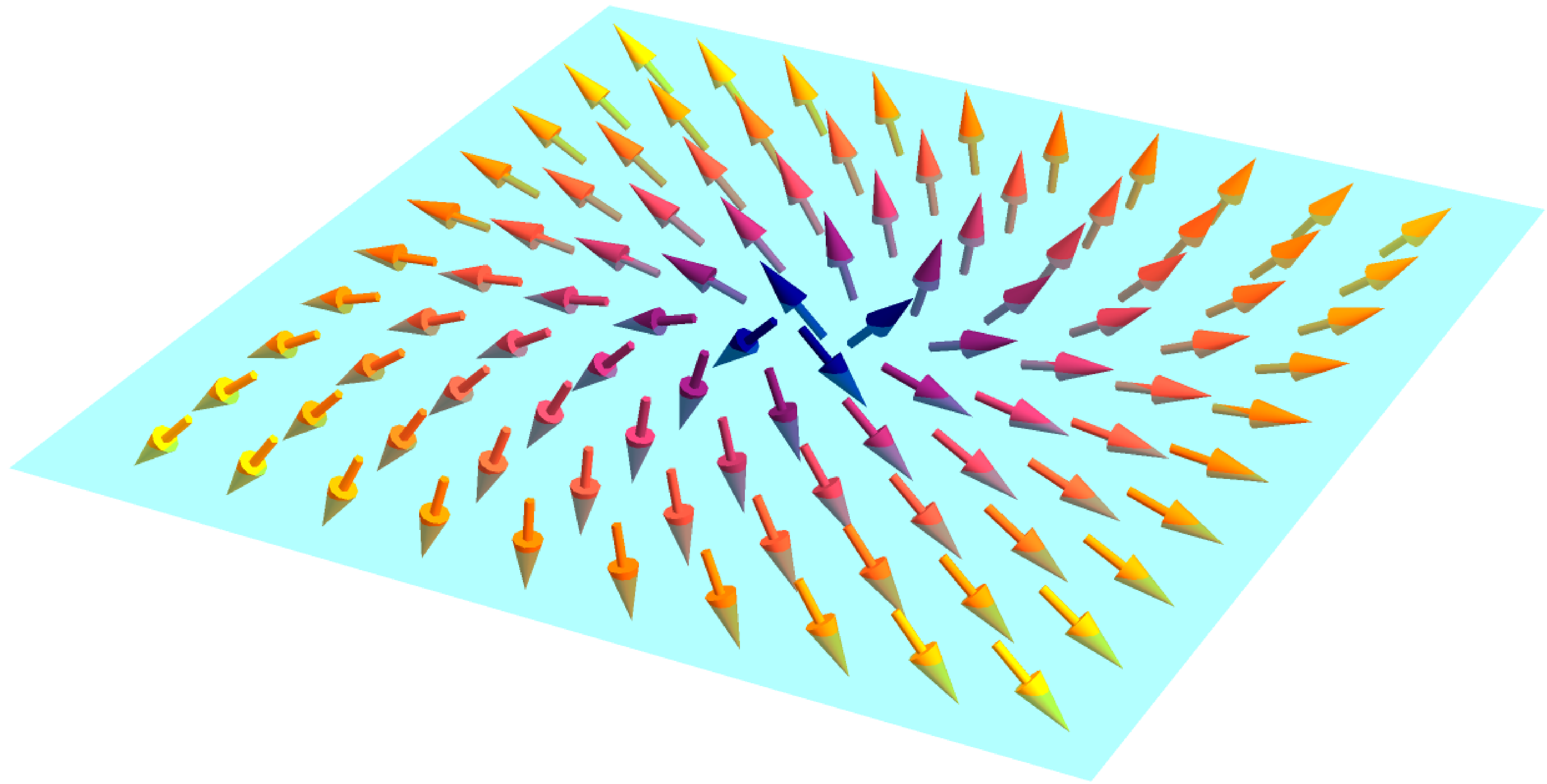
- **More general solution :**

$$\Sigma = g\Sigma_0 g^\dagger = \exp(\mathrm{i}\theta g\tau_3 g^\dagger)$$

- **SSB of SU(2)$_V$ $\to$ U(1)**

- **Σ_0 is invariant under** $g = \mathrm{e}^{\mathrm{i}\tau_3\theta}$

S^2 moduli on the domain wall



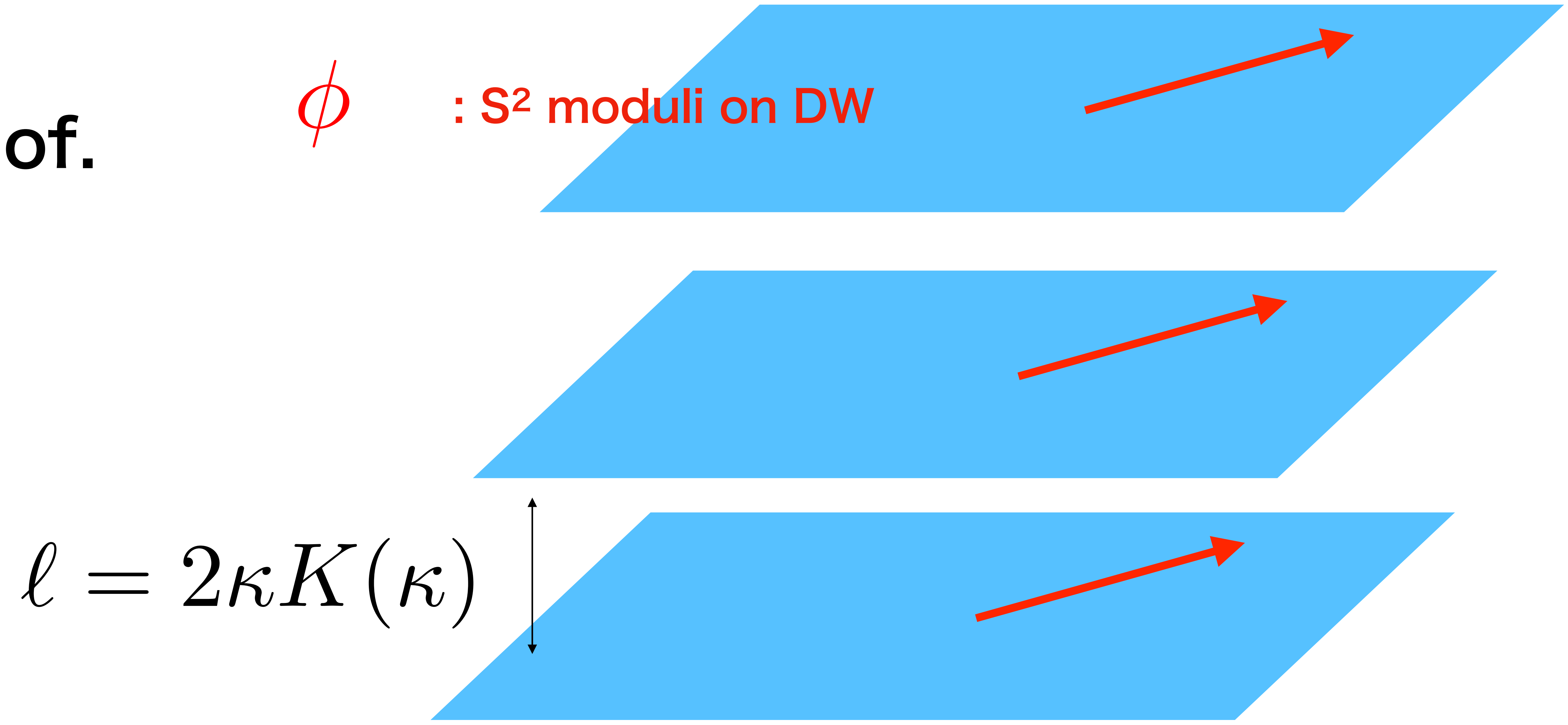
- **The collective coordinate :** $\phi \in \mathbb{C}^2\,,\qquad \phi^\dagger\phi = 1$

Nitta (2015); Eto and Nitta (2015) $\qquad g\sigma_3 g^\dagger = 2\phi\phi^\dagger - 1$

EFT of the DW

- Construct DW world effective theory via the moduli approximation.
- This EFT identifies S^2 moduli $\boldsymbol{\phi}$ as dof.
- Promote the moduli to a field on 2+1 dim world volume

$$\phi \to \phi(x^\alpha)\,, \quad (\alpha = 0, 1, 2)$$



- Integrating over the codimension z :

$$\mathcal{L}_{\mathrm{EFT}} = \int_{-\infty}^{\infty} \mathrm{d}z \, \underline{\mathcal{L}_{\mathrm{ChPT}} + \mathcal{L}_{\mathrm{B}}}$$

Substitution

$$\Sigma = \exp(2\mathrm{i}\theta\phi\phi^\dagger)u^{-\mathrm{i}\chi_3^{\mathrm{CSL}}} \quad \chi_3^{\mathrm{CSL}} = 2\mathrm{am}\left(\frac{m_\pi z}{\kappa}, \kappa\right) + \pi$$

EFT for S² moduli

- **Effective Lagrangian :** $\mathcal{L}_{\rm EFT} = \mathcal{L}_{\rm const} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm topo}$ Eto, KN and Nitta, JHEP 12 (2023) 032
- **Kinetic term :** $\mathcal{L}_{\rm kin} = \mathcal{C}(\kappa)[(\phi^\dagger D_\alpha \phi)^2 + D^\alpha \phi^\dagger D_\alpha \phi]$
- **Topological terms :** $\mathcal{L}_{\rm topo} = -2\mu_{\rm B} q + \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$

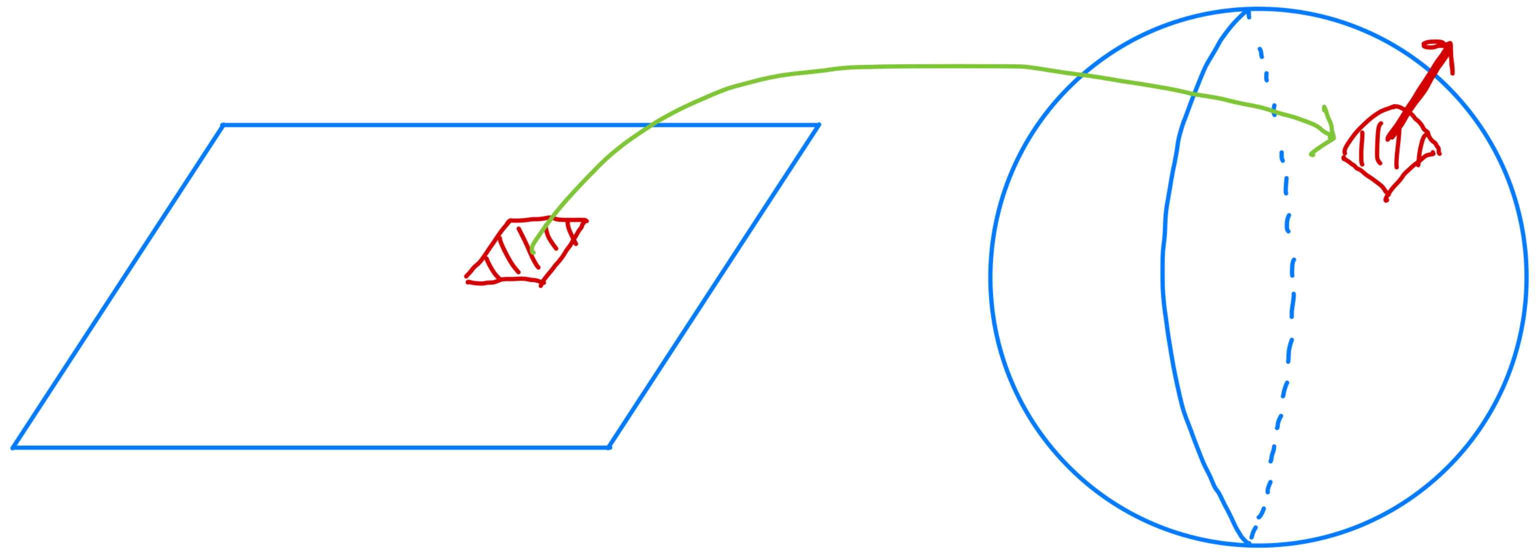
O(3) nonlinear sigma model

$$n_a \equiv \phi^\dagger \sigma_a \phi \quad |\boldsymbol{n}| = 1$$

- The red term stabilizes the configuration with finite k!

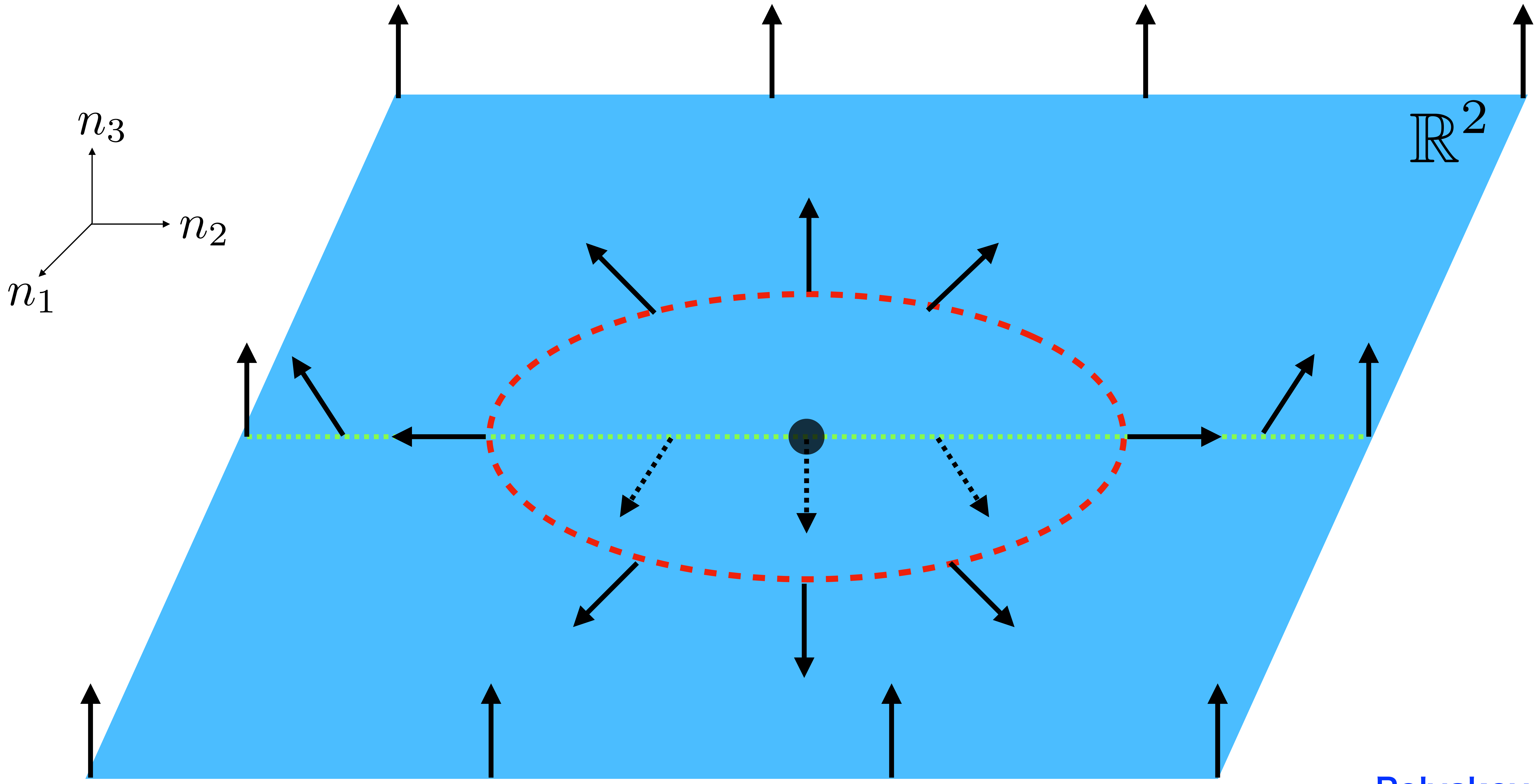
π_2(S²) topological charge (counting how many times xy plane covers S² moduli)

$$k = \int \mathrm{d}^2 x\, q$$
$$= \frac{1}{4\pi}\int \boldsymbol{n}\cdot\left(\frac{\partial \boldsymbol{n}}{\partial x}\mathrm{d}x \times \frac{\partial \boldsymbol{n}}{\partial y}\mathrm{d}y\right)$$
$$\in \mathbb{Z}$$



Baby Skyrmion

- **Configuration on DW surrounding S²** : ↑ = n_a $\quad \boldsymbol{n^2} = 1$



Polyakov and Belavin (1975)

Bogomol'nyi bound

- Baby Skyrmion naturally appears when minimizing the Hamiltonian.

$$\mathcal{H}_{\rm DW} = \frac{\mathcal{C}(\kappa)}{4}\partial_i \boldsymbol{n}\cdot\partial_i \boldsymbol{n} + 2\mu_{\rm B} q - \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$$

- Completing the square of the kinetic term is useful!

$$(\partial_i \boldsymbol{n})^2 = \frac{1}{2}\underbrace{(\partial_i \boldsymbol{n} \pm \epsilon_{ij}\boldsymbol{n}\times\partial_j \boldsymbol{n})^2}_{=0} \pm 8\pi q \geq \pm 8\pi q$$

→ BPS equation → Baby Skyrmion!

- Total energy :

$$E_{\rm DW} \geq \underbrace{2\pi\mathcal{C}(\kappa)|k| + 2\mu_{\rm B} k}_{} - \underbrace{\frac{e\mu_{\rm B}}{2\pi}\int d^2x\,\epsilon^{03jk}\partial_j[A_k(1-n_3)]}_{}$$

The total energy is negative $\mu > \mu_c$, and baby Skyrmion appears in the ground state!

Some constraints on the lump

Eto, KN and Nitta, JHEP 12 (2023) 032

Constraint on baby Skyrmion

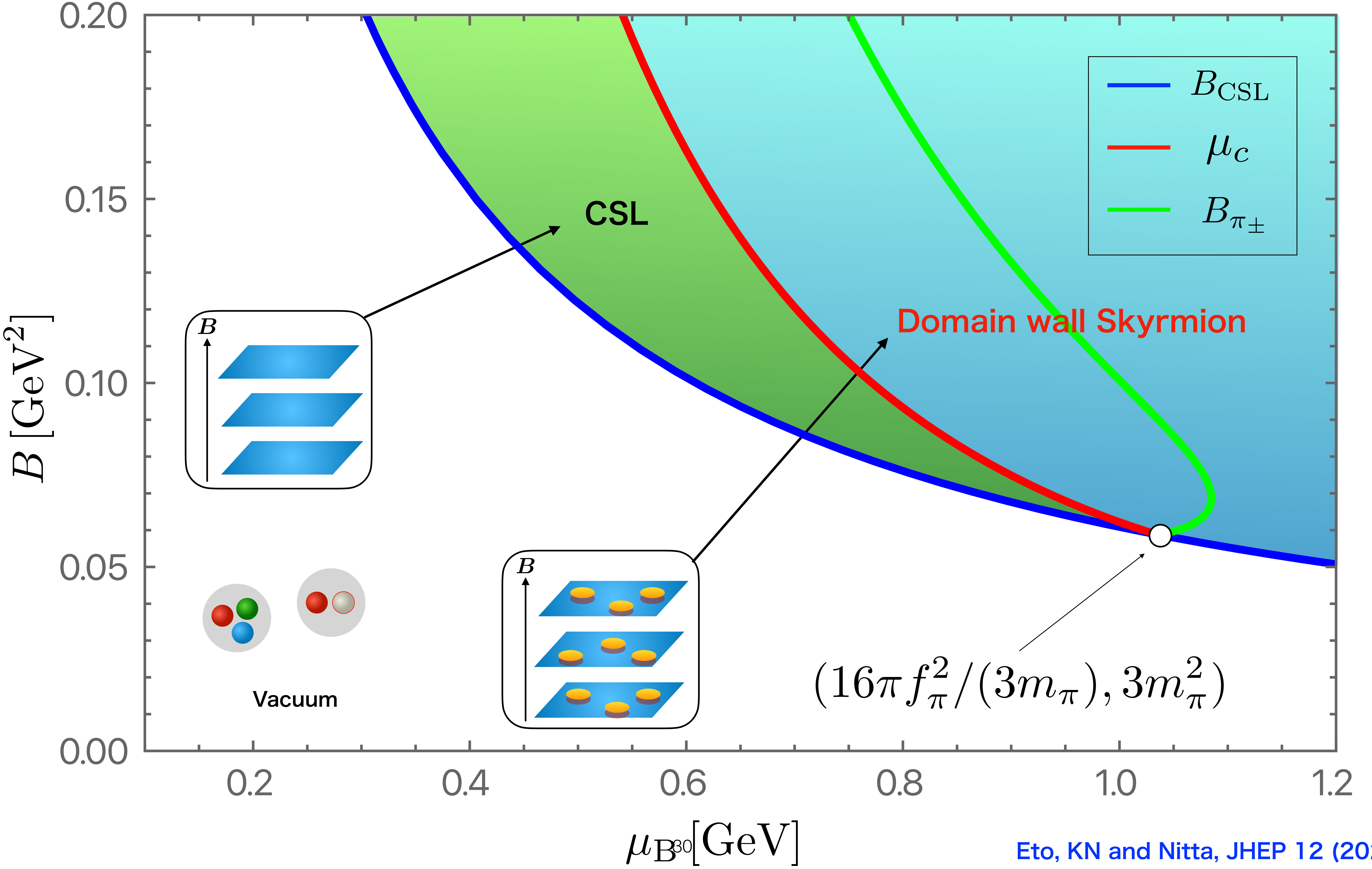
- **k anti-Baby Skyrmion solution:** $n_3 = \frac{1-|f|^2}{1+|f|^2}\,,\; f = \frac{b_{k-1}\bar{w}^{k-1} + \cdots + b_0}{\bar{w}^k + a_{k-1}\bar{w}^{k-1} + \cdots + a_0}$

- **$E_{\rm DWSk}$, for the k anti-baby Skyrmion:** $E_{\rm DWSk} = \underline{2\pi C(\kappa)|k| - 2\mu_{\rm B}|k|} + e\mu_{\rm B} B|b_{k-1}|^2$

  Can it be negative here?

- **In order to minimize $E_{\rm DWSk}$, $b_{k-1}=0$.**

- **Critical baryon chemical potential:** $\mu_{\rm B} \geq \mu_{\rm c} = \pi C(\kappa)$ Eto, KN and Nitta, JHEP 12 (2023) 032



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Eto, KN and Nitta, JHEP 12 (2023) 032

Summary

- We have to include the minimal coupling of pions to baryons via Skyrmions.

- At $B>B_c$, the parallel stack of π_0 DWs is energetically stable.

- At $\mu>\mu_c$, baby Skyrmion appears on π_0 DWs.

Future direction

- **DWSk in QCD-like theory (two-color QCD)**
  - Two-color QCD with finite baryon chemical potential and magnetic field has no sign problem.
  - CSL is QCD-like theory has been considered. Brauner, Filions and Kolesova (2019)

- **DWSk in lattice gauge theory**
  - Monte-Carlo simulation
  - Strong-coupling expansion for calculating free energy

    See also Nishida (2004) and Nishida, Fukushima and Hatsuda (2004)

Thank you for your attention!

Back up

Elliptic integrals and functions

- **The elliptic integral of the first kind :** $k' = \sqrt{1-k^2}$

$$K(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \frac{1}{\sqrt{1-k^2\sin^2\theta}} \simeq \ln\frac{4}{k'^2} + \frac{k'^2}{4}\left(\ln\frac{4}{k'^2} - 1\right)$$

- **The elliptic integral of the second kind :**

$$E(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \sqrt{1-k^2\sin^2\theta} \simeq 1 + \frac{k'^2}{2}\left(\ln\frac{4}{k'^2} - \frac{1}{2}\right)$$

EOM of the fluctuations

- **Fluctuation around the CSL background :**

$$\omega^2\pi_+ = \underbrace{\left[-\partial_x^2 + B^2\left(x - \frac{p_y}{B}\right)^2\right]}_{\text{Giving the Landau quantization}}\pi_+ + (\partial_z^2 + 2\mathrm{i}\partial_z + m_\pi^2 \mathrm{e}^{\mathrm{i}\phi_3})\pi_+$$

- **Chiral limit :** $\omega^2 = p_z^2 - \underbrace{\frac{\mu_B B p_z}{2\pi^2 f_\pi^2}}_{\text{Deducing the energy!}} + (2n+1)B$

- **ω^2<0 :** $B_{\pi_\pm} = \dfrac{16\pi^4 f_\pi^2}{\mu_B^2}$ Brauner and Yamamoto (2017)