New perspectives, connections, and generalizations of Efimov physics

Chris H. Greene, Purdue University

Michael Higgins
(now a postdoc at Purdue)

These two former PhD students in my group have done the heavy lifting on the topics I will discuss today

Yu-Hsin Chen
(now in a faculty position in Taiwan)

Thanks to the NSF for support!

Strategy of the adiabatic hyperspherical representation: convert the partial differential Schroedinger equation into an infinite set of coupled ordinary differential equations:

First, we set:

$$\psi = R^{(3(N-1)-1)/2}\Psi$$

(in order to eliminate 1st order derivatives in K.E. operator)

To solve:

$$\left[-\frac{1}{2\mu}\frac{\partial^2}{\partial R^2}+\frac{\Lambda^2}{2\mu R^2}+V(R,\theta,\varphi)\right]\psi_E = E\psi_E$$

Now solve the fixed-R Schroedinger equation, for eigenvalues $U_n(R)$:

$$\left[\frac{\Lambda^2}{2\mu R^2}+\frac{15}{8\mu R^2}+V(R,\theta,\varphi)\right]\Phi_\nu(R;\Omega) = U_\nu(R)\Phi_\nu(R;\Omega)$$

Next expand the desired solution into the complete set of eigenfunctions

$$\psi_E(R,\Omega) = \sum_\nu F_{\nu E}(R)\Phi_\nu(R;\Omega)$$

And the original T.I.S.Eqn. is transformed into the following set (coupled ODEs) which can be truncated on physical grounds, with the eigenvalues interpretable as adiabatic potential curves, in the Born-Oppenheimer sense.

$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2}+U_\nu(R)\right]F_{\nu E}(R)-\frac{1}{2\mu}\sum_{\nu'}\left[2P_{\nu\nu'}(R)\frac{d}{dR}+Q_{\nu\nu'}(R)\right]F_{\nu' E}(R) = EF_{\nu E}(R)$$

Different calculation methods of the adiabatic hyperspherical potential curves and nonadiabatic couplings:

Method 1. Analytical treatment (3 bodies with zero-range interactions), using the hyperangular Green's function (Rittenhouse, Mehta, CHG, PRA 82 022706

Green's functions and the adiabatic hyperspherical method

For a system of N particles, x_i denote the d = 3N − 3 Cartesian components needed to specify the relative positions of the N particles (with d-1 hyperangles here)

$$\sqrt{\mu}R = \sqrt{\sum_{i=1}^{d} \mu_i x_i^2}$$

$$\Lambda^2 = -\sum_{i<j} \Lambda_{ij}^2,$$

$$\Psi(R,\Omega) = \sum R^{-(d-1)/2} F_n(R) \Phi_n(R;\Omega)$$

$$\left[\frac{\hbar^2}{2\mu}\frac{\Lambda^2}{R^2} + V(R,\Omega)\right]\Phi_n(R;\Omega) = u_n(R)\Phi_n(R;\Omega)$$

$$\Lambda_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$$

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dR^2} - \frac{(d-3)(d-1)}{4R^2}\right) + u_n(R)\right] F_n(R) - \frac{\hbar^2}{2\mu}\sum_m \left[2P_{nm}\frac{d}{dR} + Q_{nm}\right] F_m(R) = E F_n(R).$$

$$P_{mn} = \left\langle \Phi_m(R;\Omega) \middle| \frac{\partial}{\partial R} \Phi_n(R;\Omega) \right\rangle,$$

$$Q_{mn} = \left\langle \Phi_m(R;\Omega) \middle| \frac{\partial^2}{\partial R^2} \Phi_n(R;\Omega) \right\rangle$$

Some basic properties of adiabatic potential energy curves in hyperspherical coordinates:

1. The hyperangular operator Λ^2 is Hermitian and has R-independent eigenvalues, which means that the diagonal potentials $U_n(R)$ for the NONINTERACTING system (V=0) are all known analytically,

$$U_\nu(R) \to \frac{\ell_{\rm NI}(\ell_{\rm NI}+1)\hbar^2}{2\mu R^2}$$

e.g. for 3 bosons, the lowest value of $\ell_{\rm NI}$ is 3/2, whereas for 4 bosons, the lowest value of $\ell_{\rm NI}$ is 3. This is important for discerning threshold laws of transition matrix elements to and/or from the N-body continuum states, as in recombination, i.e. for any short range dominated process,

$$T_{f,i} \propto k_f^{\ell_f+1/2} k_i^{\ell_i+1/2}$$

2. For every possible fragmentation threshold of the N-body system allowed for the symmetry being considered, there must be at least one potential curve converging to its energy at R → infinity

3. When an Efimov effect is present, at least one potential curve converging to E=0 is negative, meaning that

$$\ell_{\rm eff} \equiv -\frac{1}{2} + i s_0$$

4. For s-wave δ-function interactions in 3D with finite *a*, the asymptotic potentials converging to the N-body continuum have the form:

$$\text{as } R \to \infty, U_{\rm min}(R) \to \frac{\ell_{\rm NI}(\ell_{\rm NI}+1)\hbar^2}{2\mu R^2} + C\frac{a}{R^3}$$

Implications of the attractive a/ρ³ potential at long range

One can readily derive (e.g. using the Born approximation) that the limiting low energy phaseshift in such a potential of the form,

$$u_0(\rho) \to \frac{\hbar^2}{2\mu}\left(\frac{l_{\rm eff}(l_{\rm eff}+1)}{\rho^2} + C\frac{a}{\rho^3}\right)$$

is: $\underset{k \to 0}{\delta \to} -Cak/(2l_{\rm eff} + 2l_{\rm eff}{}^2)$

So the phaseshift is linear in k for ALL nonzero values of the centrifugal potential, but any transition matrix element obeys the ordinary Wigner threshold law, namely:

$$T_{f,i} \propto k_f^{\ell_f+1/2} k_i^{\ell_i+1/2}$$

Some other interesting special cases for 3 or 4 fermions :

N, L^π (spin state)	$\ell_{\rm eff}$	$\ell_{\rm eff}$ (unitarity)
3,0+, (spins up,up,up)	15/2	1.00 (V^p infinite scatt. Vol.)
3,1+, (spins up,up,up)	7/2	1.00 (V^p infinite)
3,1-, (spins up,up,up)	9/2	0.011 (V^p infinite)
3,1+, (up,up,down)	7/2	1.00 (V^p infinite)
3,1- , (up,up,down)	5/2	0.00 (V^p infinite)

Yu-Hsin Chen & CHG, PRA 2022 (3 fermions, p-wave unitarity)

4,0+, (up,up,down,down)	5	2.01 (*a* infinite)

PRA 2022 Michael Higgins 3 or 4 fermions

Recall that in the ordinary Efimov effect for 3 bosons or 3 different spin fermions, the following values hold:

$\ell_{\rm eff}$ = 3/2 $\qquad$ $\ell_{\rm eff}$ (unitarity, a) = -1/2 + i ←(approximately)

i.e. $\ell_{\rm eff}(\ell_{\rm eff}+1)$ ~ -5/4

Note that $K(N\text{-body recomb}) \propto k^{6+2\ell_{\rm eff}-3N}$

L^{Π}	$l_{e,\nu}$									Case
0^+	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	23/2	non-interacting
	1.666	**4.627**	**6.614**	15/2	**8.332**	19/2	**10.562**	23/2	23/2	↑↓ at s-wave unitary limit and ↑↑ non-interacting
	$\underline{1}$	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	↑↑ at p-wave unitary limit and ↑↓ non-interacting
	$\underline{1}$	**1.666**	**4.627**	**6.614**	15/2	**8.332**	19/2	**10.562**	23/2	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit
1^+	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	23/2	non-interacting
	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	23/2	↑↓ at s-wave unitary limit and ↑↑ non-interacting
	$\underline{1}$	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	↑↑ at p-wave unitary limit and ↑↓ non-interacting
	$\underline{1}$	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit
1^-	5/2	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2	non-interacting
	1.272	**3.858**	9/2	**5.216**	13/2	13/2	**7.553**	17/2	17/2	↑↓ at s-wave unitary limit and ↑↑ non-interacting
	$\underline{0}$	$\underline{2}$	5/2	9/2	9/2	13/2	13/2	13/2	17/2	↑↑ at p-wave unitary limit and ↑↓ non-interacting
	$\underline{0}$	**1.272**	$\underline{2}$	**3.858**	9/2	**5.216**	13/2	13/2	**7.553**	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit
2^-	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2	21/2	non-interacting
	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2	21/2	↑↓ at s-wave unitary limit and ↑↑ non-interacting
	$\underline{2}$	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2	↑↑ at p-wave unitary limit and ↑↓ non-interacting
	$\underline{2}$	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit

TABLE II. Comparison of the 1st to 9th $l_{e,\nu}$ values for trimers consisting of two spin-up and one spin-down fermion (↓↑↑) with the non-interacting values of $l_{e,\nu}$. Cases considered involve different spin fermions that are either at s-wave unitarity or else are noninteracting as are labeled in the Table. Other cases show the same spin fermions either at p-wave unitarity or noninteracting. The bold and underlined numbers correspond to the $l_{e,\nu}$ values of the s-wave unitary channel and the p-wave unitary channel, respectively.

Unitarity reductions of the centrifugal barrier constant l_e for Fermionic 3-body systems of different symmetries

Yu-Hsin Chen & CHG
arXiv:2408.08993
Universal trimers with p-wave interactions and the faux-Efimov effect

See also 2022 PRA

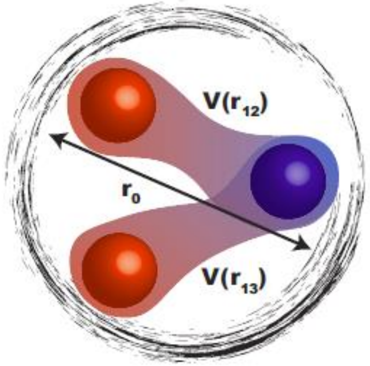
For some symmetries with p-wave interactions at unitarity a faux-Efimov effect, discussed in my tutorial lectures at some length

Nonresonant Density of States Enhancement for Few-Neutron Systems

Chris Greene, Purdue University

with Michael Higgins, Alejandro Kievsky and Michele Viviani
(on the 3n and 4n systems)

and on the 3 unitary fermions,
with Yu-Hsin Chen



Outline for Few Neutrons

1. Theoretical search for a low energy bound or resonant trineutron or tetraneutron, and connections with Efimov physics and universality, see PRL 125, 052501 (2020)

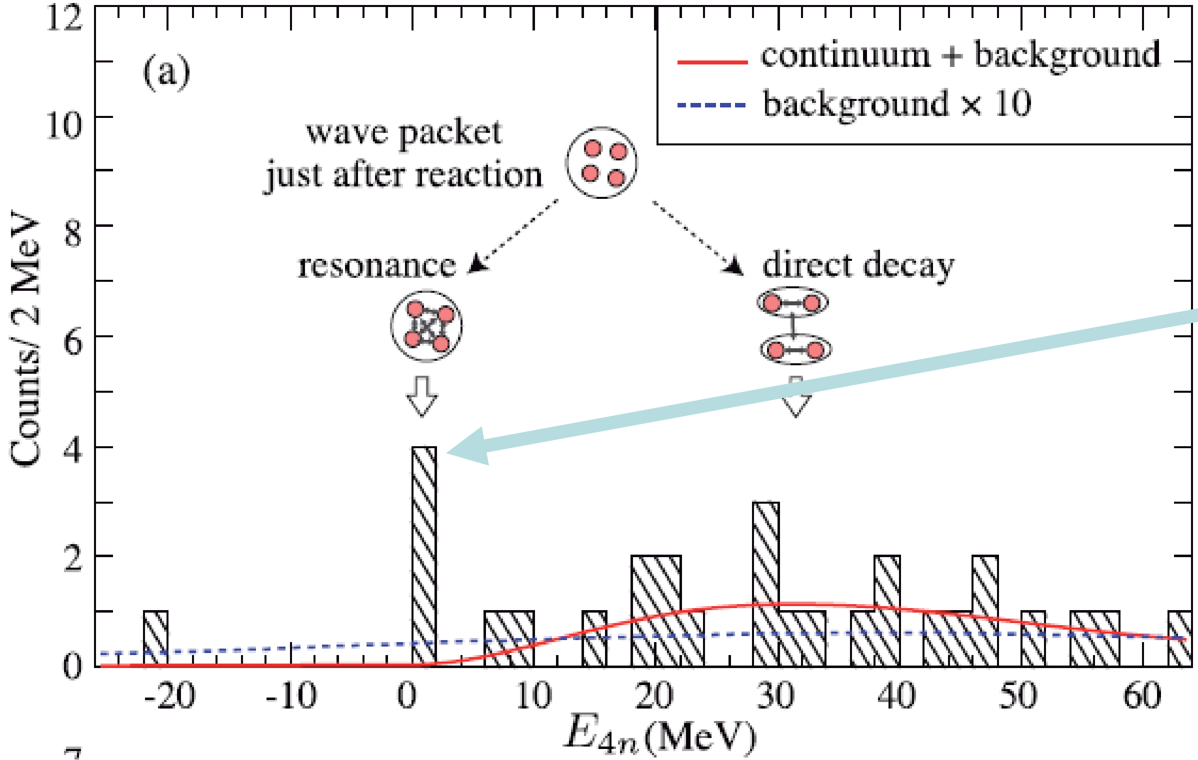
Further elaborations were subsequently published in: Phys. Rev. C 103, 024004 (2021)

Results from the 3n, 4n experimental literature:

Expt: a 4n candidate published in PRL 116, 052501 (2016), Kisamori et al.
conclusion: its energy is $0.83 \pm 0.65(\mathrm{stat}) \pm 1.25(\mathrm{syst})$ MeV

And a Nature News & Views by Bertulani & Zelevinsky, > 2000 page views

This experiment measured the reaction ^{4}He+^{8}He → ^{8}Be+4n
…. And observed a surplus of very low energy events, interpreted as a tetraneutron resonance (or even a bound state couldn't be ruled out)



Surplus of 4n events are observed at low energy!

And, in a more recent experiment, Physics Letters B **824**(2022)136799:

Indications for a bound tetraneutron

Reaction studied: $^{7}Li + ^{7}Li \rightarrow ^{10}C + 4n$

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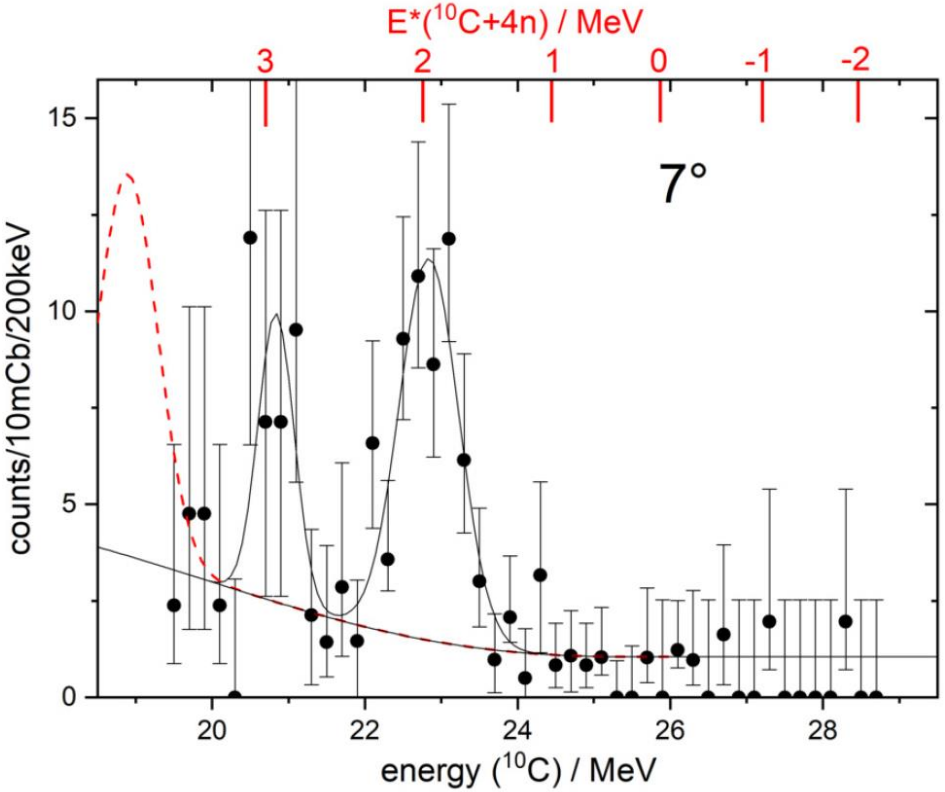
"…Therefore, we favor the interpretation that this peak corresponds to ^{10}C in the first excited state at 3.354 MeV and a tetraneutron with a binding energy of +0.42 ±0.16MeV."



Using the reaction $^{7}Li(^{7}Li,^{10}C)$ we tried to populate states in the tetraneutron. A peak in the energy spectrum of identified ^{10}C, which we cannot attribute to a reaction with any other of the target components, corresponds to an excitation of the ^{10}C+4n system of 2.93 ± 0.16 MeV. Under different kinematic conditions an equivalent peak was observed. For a binding energy of the tetraneutron of -2.93 MeV a much larger width than the observed upper limit of $\Gamma < 0.24$ MeV (mainly due to experimental spread) is expected. Therefore, we favor the interpretation that this peak corresponds to ^{10}C in the first excited state at 3.354 MeV and a tetraneutron with a binding energy of $+0.42 \pm 0.16$ MeV.

Observation of a correlated free four-neutron system

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Here we report on the observation of a resonance-like structure near threshold in the four-neutron system that is consistent with a quasi-bound tetraneutron state existing for a very short time.

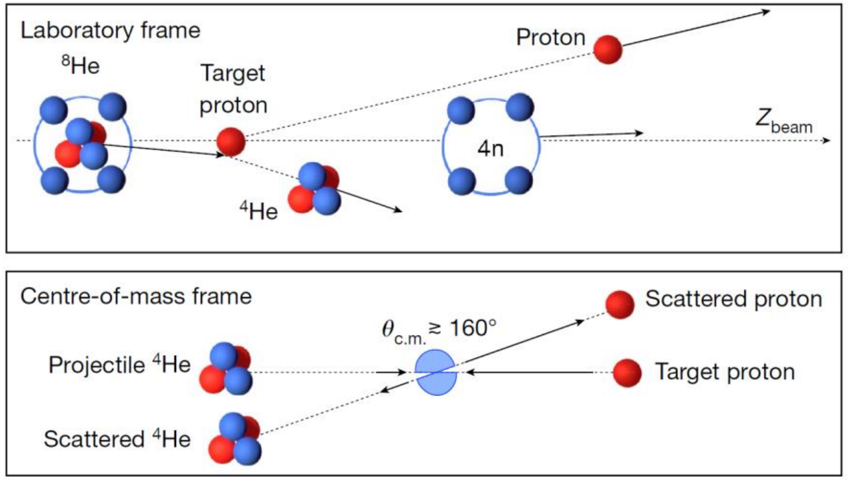
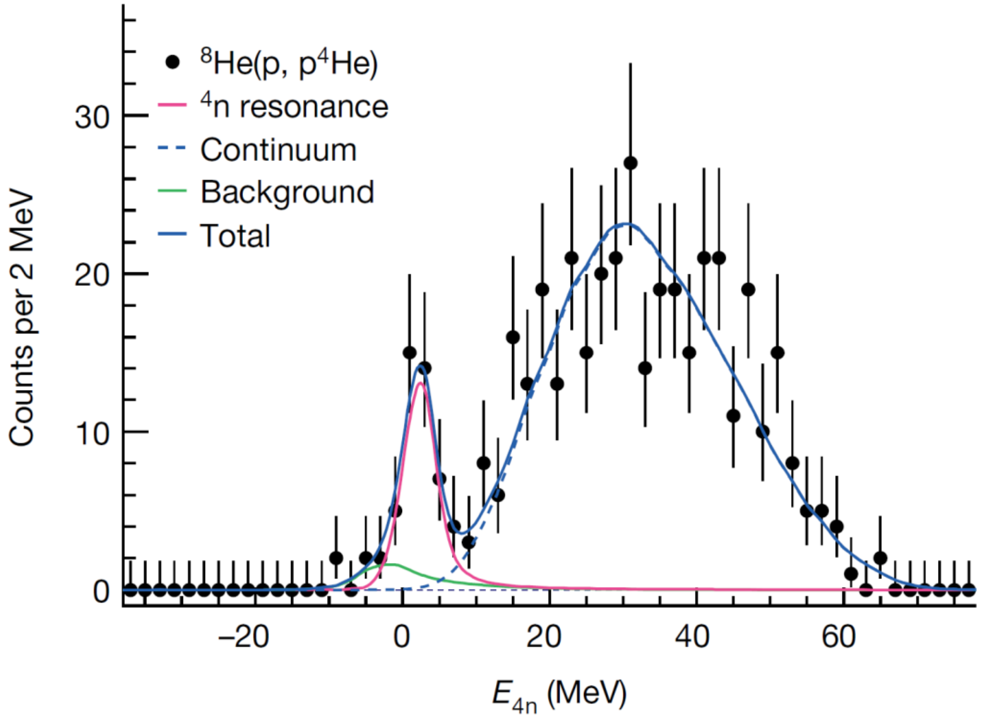


Fig. 1 | Schematic illustration of the quasi-elastic reaction investigated in this work. Top: quasi-elastic scattering of the ^{4}He core from a ^{8}He projectile off a proton target in the laboratory frame. The length of the arrows represents the momentum per nucleon (the velocity) of the incoming and outgoing particles. Z_{beam} is the beam axis. Bottom: the equivalent p–^{4}He elastic scattering in their centre-of-mass frame, where we consider reactions at backward angles close to 180°, $\theta_{c.m.} \gtrsim 160°$. In this frame, the momentum of the proton balances that of the ^{4}He, $\mathbf{P}_p = -\mathbf{P}_{^4He}$, that is, the proton is four times faster than the ^{4}He.



Shortly after the publication of this experimental result by Kisamori et al. in 2016, many nuclear theory groups tackled this problem with renewed excitement, trying to figure out whether our current understanding of nuclear forces is consistent with the possibility of a 4-neutron resonance existing at low energy below 1 MeV.

The answers from the strongest theory groups?

Hiyama , Lazauskas, Carbonell, Kamimura : NO resonance can exist
Gandolfi, Hammer, Klos, Lynn, Schwenk : a resonance could exist
Fossez, Rotureau, Michel, Ploszajczak : a resonance should exist
Shirokov,Papadimitriou,Mazur,Mazur, Roth,Vary : a resonance should exist
Deltuva: NO resonance can exist

Intrigued by this controversy, Michael Higgins, myself, and the Pisa group of Alejandro Kievsky and Michele Viviani decided to address this problem using our adiabatic hyperspherical toolkit – it took us 3 years!

First: a brief review of recent 3n, 4n theory

Theory: Hiyama, Lazauskas, Carbonell, Kamimura 2016 Phys. Rev. C.
conclusion: "…a remarkably attractive 3N force would be required…" NO!

Gandolfi, Hammer, Klos, Lynn, Schwenk *Phys. Rev. Lett. 118, 232501 2017* YES!
conclusion: a three-neutron resonance exists below a four-neutron resonance in nature and is potentially measurable

Fossez, Rotureau, Michel, Ploszajczak *Phys. Rev. Lett. 119, 032501 (2017)* YES!
conclusion: while the energy (4n) …may be compatible with expt… its width must be larger than the reported upper limit ➔ (probably) …reaction process too short to form a nucleus

Shirokov,Papadimitriou,Mazur,Mazur, Roth,Vary *Phys. Rev. Lett. 117, 182502* (2016) conclusion: 4n resonance, E=0.8 MeV, $\Gamma = 1.4$ MeV YES!

Tetraneutron: Rigorous continuum calculation, A.Deltuva in Physics Letters B 782(2018) 238–241 "... This indicates the absence of an observable 4n resonance..." NO!

Tri-neutron: Three-neutron resonance study using transition operators A. Deltuva, PRC 97, 034001 (2018) "...There are no physically observable three-neutron resonant states consistent with presently accepted interaction models" NO!

Our main theoretical tool: formulate the problem in hyperspherical coordinates, treating the hyperradius R adiabatically

The hyperradius R (squared) is a coordinate proportional to total moment of inertia of any N-particle system, i.e.:

$$R^2 = \frac{1}{M}\sum_i m_i r_i^2 \quad \text{also } \rho^2$$

Here r_i is the distance of the i-th particle from the center-of-mass. All other coordinates of the system are 3N-4 hyperangles.

And then the rest of the problem comes down to calculating energy levels as a function of R, which we call "hyperspherical potential curves", and their mutual couplings, which can then be used to compute bound state and resonance properties, scattering and photoabsorption behavior, nonperturbatively

This follows the formulation of the N-body problem in the adiabatic hyperspherical representation, as pioneered by Macek, Fano, Lin, Klar, and others

Strategy of the adiabatic hyperspherical representation: **FOR ANY NUMBER OF PARTICLES**, convert the partial differential Schroedinger equation into an infinite set of coupled **ordinary** differential equations:

To solve:

$$\left[-\frac{1}{2\mu}\frac{\partial^2}{\partial R^2}+\frac{\Lambda^2}{2\mu R^2}+V(R,\theta,\varphi)\right]\psi_E=E\psi_E$$

First solve the fixed-R Schroedinger equation, for eigenvalues $U_n(R)$:

$$\left[\frac{\Lambda^2}{2\mu R^2}+\frac{15}{8\mu R^2}+V(R,\theta,\varphi)\right]\Phi_\nu(R;\Omega)=U_\nu(R)\Phi_\nu(R;\Omega)$$

Next expand the desired solution into the complete set of hyperangle eigenfunctions with unknowns F(R)

$$\psi_E(R,\Omega)=\sum_\nu F_{\nu E}(R)\Phi_\nu(R;\Omega)$$

And the original T.I.S.Eqn. is transformed into the following set which can be truncated on physical grounds, with the eigenvalues interpretable as adiabatic potential curves, in the Born-Oppenheimer sense.

$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2}+U_\nu(R)\right]F_{\nu E}(R)-\frac{1}{2\mu}\sum_{\nu'}\left[2P_{\nu\nu'}(R)\frac{d}{dR}+Q_{\nu\nu'}(R)\right]F_{\nu' E}(R)=EF_{\nu E}(R)$$

Notes

- Various methods can be used to compute the needed potential curves and couplings: diagonalization in a basis set of hyperspherical harmonics, or correlated Gaussians, or Monte Carlo techniques, etc.

- A theorem exists about the truncation to a single potential curve, i.e. the adiabatic approximation, namely:

$\rightarrow$If all diagonal and nondiagonal couplings are neglected, the lowest potential energy curve alone gives a rigorous LOWER BOUND on the ground state energy of the system

$\rightarrow$If only the diagonal nonadiabatic term is included and all other couplings are neglected, the lowest potential gives a rigorous UPPER BOUND on the ground state energy

$$W_{\nu,\nu'}(\rho) = -\frac{\hbar^2}{2\mu}\left(\langle \Phi_\nu | \frac{\partial}{\partial\rho}\Phi_{\nu'}\rangle \frac{\partial}{\partial\rho} + \langle \Phi_\nu | \frac{\partial^2}{\partial\rho^2}\Phi_{\nu'}\rangle\right) + U_\nu(R)\ \delta_{\nu,\nu'}$$

An important aside: The d-dimensional Laplacian operator is:

$$-\nabla^2 = -\frac{1}{R^{d-1}}\partial_R R^{d-1}\partial_R + \frac{\hat{K}^2}{R^2}$$

This Laplacian operator acts on the full wavefunction, so like one normally does in d=3, we can rescale the radial wavefunction, i.e. set

$$\Psi(R,\Omega) = \frac{1}{R^{\frac{d-1}{2}}}\psi(R,\Omega) \rightarrow \frac{1}{R^{\frac{d-1}{2}}}F(R)\Phi_\nu(R;\Omega)$$

$$-\nabla^2\Psi \rightarrow \frac{1}{R^{\frac{d-1}{2}}}\left(-\frac{\partial^2}{\partial R^2} + \frac{\hat{K}^2}{R^2} + \frac{(d-1)(d-3)}{4R^2}\right)F(R)\Phi_\nu(R;\Omega)$$

Nonadiabatic coupling terms

$$\left[\frac{-1}{2\mu_N}\frac{\partial^2}{\partial R^2} + W_i(R) - E\right]F_i + \boxed{\sum_{f\neq i} V_{if}(R)F_f} = 0$$

$$W_\lambda(R) \rightarrow \frac{l_e(l_e+1)}{2\mu_N R^2} \quad \text{with} \quad l_e = (2\lambda + d - 3)/2.$$

Note: d= dimension of the relative Jacobi coordinates of the system, i.e. d=6 for 3 particles, d=9 for 4 particles, etc. d=3N-3

For the 4n problem, d=9, $\lambda_{\min} = 2$, so for this symmetry, the centrifugal barrier at large R in the lowest channel is $l_e = 5$

Some results about N-body elastic scattering, from Mehta et al., PRL 103, 153201 (2009)

$$\sigma^{\text{elastic}}(d \to 3N-3) = N_p\left(\frac{2\pi}{k}\right)^{d-1}\left(\frac{\Gamma(d/2)}{2\pi^{d/2}}\right)\sum_{K\nu,K'\nu'}\left|S_{K\nu,K'\nu'} - \delta_{KK'}\delta_{\nu\nu'}\right|^2$$

and this simplifies, if a single scattering matrix eigenchannel dominates, to:

$$\sigma^{\text{elastic}}(d \to 3N-3) = N_p\left(\frac{2\pi}{k}\right)^{d-1}\left(\frac{4\Gamma(d/2)}{2\pi^{d/2}}\right)\sin^2\delta_\alpha$$

This quantity is in principle an observable, which would be relevant for the thermalization of a gas of neutrons that are out of thermal equilibrium, even if it is not a typical observable that could be readily seen in any standard nuclear physics experiment today....

And an important tool for resonance analysis on the real energy axis is the collisional time delay:

$$Q^{\text{Wigner-Smith}}(E) = i\hbar S(E)\frac{dS^{(dag)}(E)}{dE} \to 2\hbar\frac{d\delta_\alpha(E)}{dE}$$

Next, consider the 4-fermion problem

which our group had worked on extensively, 10-15 years ago, e.g.:

IOP PUBLISHING JOURNAL OF PHYSICS B: ATOMIC, MOLECULAR AND OPTICAL PHYSICS

J. Phys. B: At. Mol. Opt. Phys. **44** (2011) 172001 (43pp) doi:10.1088/0953-4075/44/17/172001

TOPICAL REVIEW

The hyperspherical four-fermion problem

S T Rittenhouse[1,2], J von Stecher[1], J P D'Incao[1], N P Mehta[1,3] and C H Greene[1]

Much is known about 4 spin-1/2 fermions, since this is a few-body version of the BEC-BCS crossover problem (we treated theoretically, e.g. PRA 2009)

For the 4-neutron system, since there are no bound subsystems, this is simplest to treat in the H-type Jacobi tree:

$$\vec{\rho}_1^{\,s} = \sqrt{\frac{4^{1/3}}{2}}\left(\vec{r}_1 - \vec{r}_3\right),$$

← 2 spin up neutrons, p-wave $Y_{1m}(13)$

$$\vec{\rho}_2^{\,s} = \sqrt{\frac{4^{1/3}}{2}}\left(\vec{r}_2 - \vec{r}_4\right),$$

← 2 spin down neutrons, p-wave $Y_{1m'}(24)$

$$\vec{\rho}_3^{\,s} = \sqrt{4^{1/3}}\left(\frac{\vec{r}_1 + \vec{r}_3}{2} - \frac{\vec{r}_2 + \vec{r}_4}{2}\right)$$

← s-wave Y_{00} in the motion of the two pairs about each other

So we consider the L=0, S=0, even parity symmetry, which corresponds to K=2, 4, 6, … and the lowest channel asymptotically should have a zeroth order potential curve

$$U^{NI}(R) = \hbar^2 \frac{K(K+7)+12}{2\mu R^2} \rightarrow \frac{30\hbar^2}{2\mu R^2}$$

Considerations about the UNITARY limit (4n)

Rakshit and Blume, Phys. Rev. A 86, 062513 (2012) found that as **a-> - infinity**, the hyperspherical potentials are entirely repulsive, at $|a| >> \rho$:

$$U^{\text{unitary}}(\rho) \to \approx \frac{6\hbar^2}{2\mu\rho^2}$$

Whereas in the noninteracting limit the asymptotic potential for two spin-up and two spin-down identical fermions is known to be:

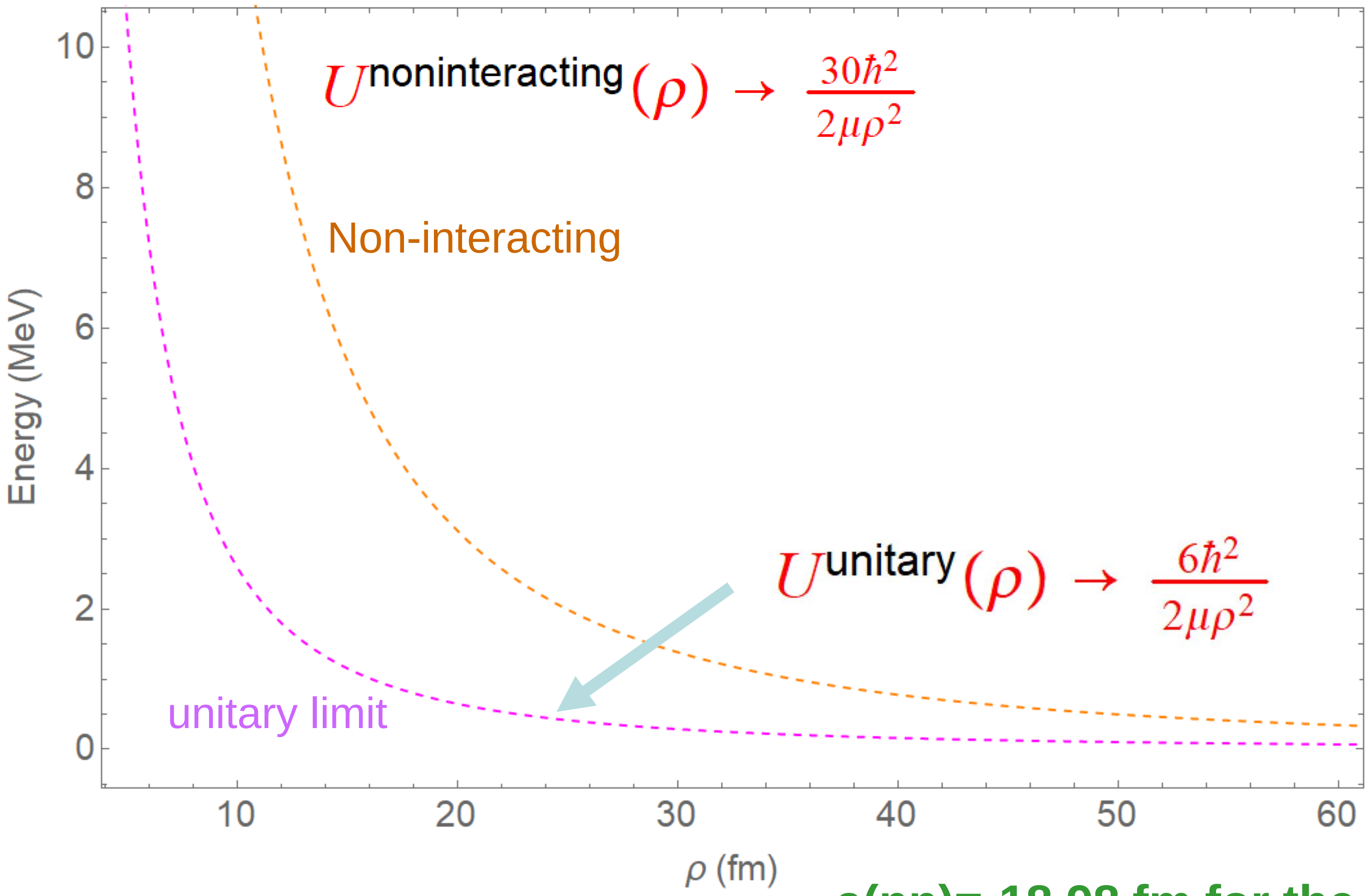
$$U^{\text{noninteracting}}(\rho) \to \frac{30\hbar^2}{2\mu\rho^2}$$

Aside: This reduction of the coefficient of the asymptotic $1/\rho^2$ potential in the unitary limit is analogous to the Efimov effect...

First Conclusion: The true potential for 4n in this symmetry is expected to be less attractive than the lower of these two potential curves, making the possibility of a bound state for this symmetry unlikely.

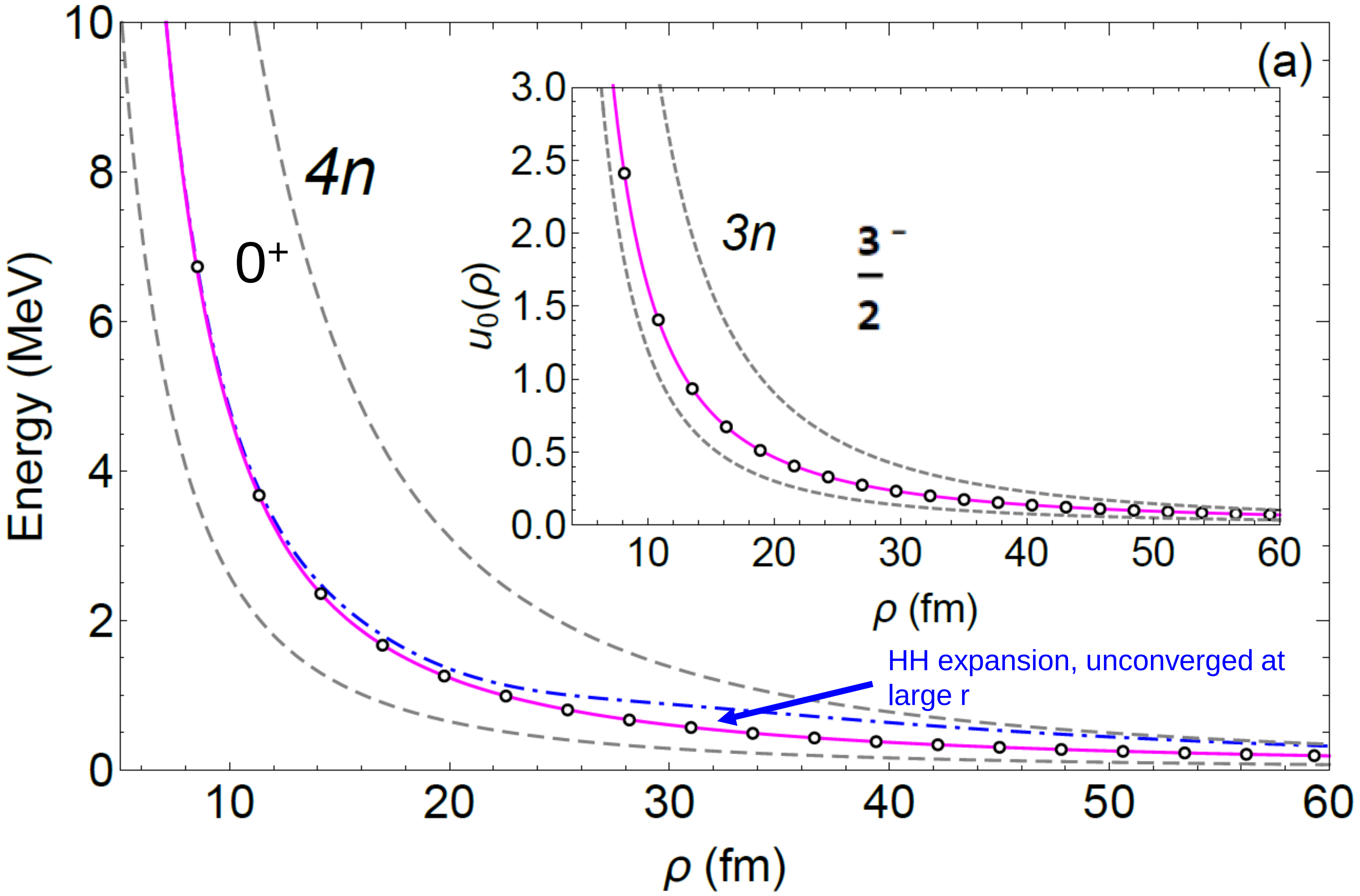
a(nn)=-18.98 fm for the Argonne AV8' potential, similar for AV18.

$J^{\pi} = 0^{+}$ Hyperspherical potentials for 4 fermions: both noninteracting and unitary limit a→-infinity



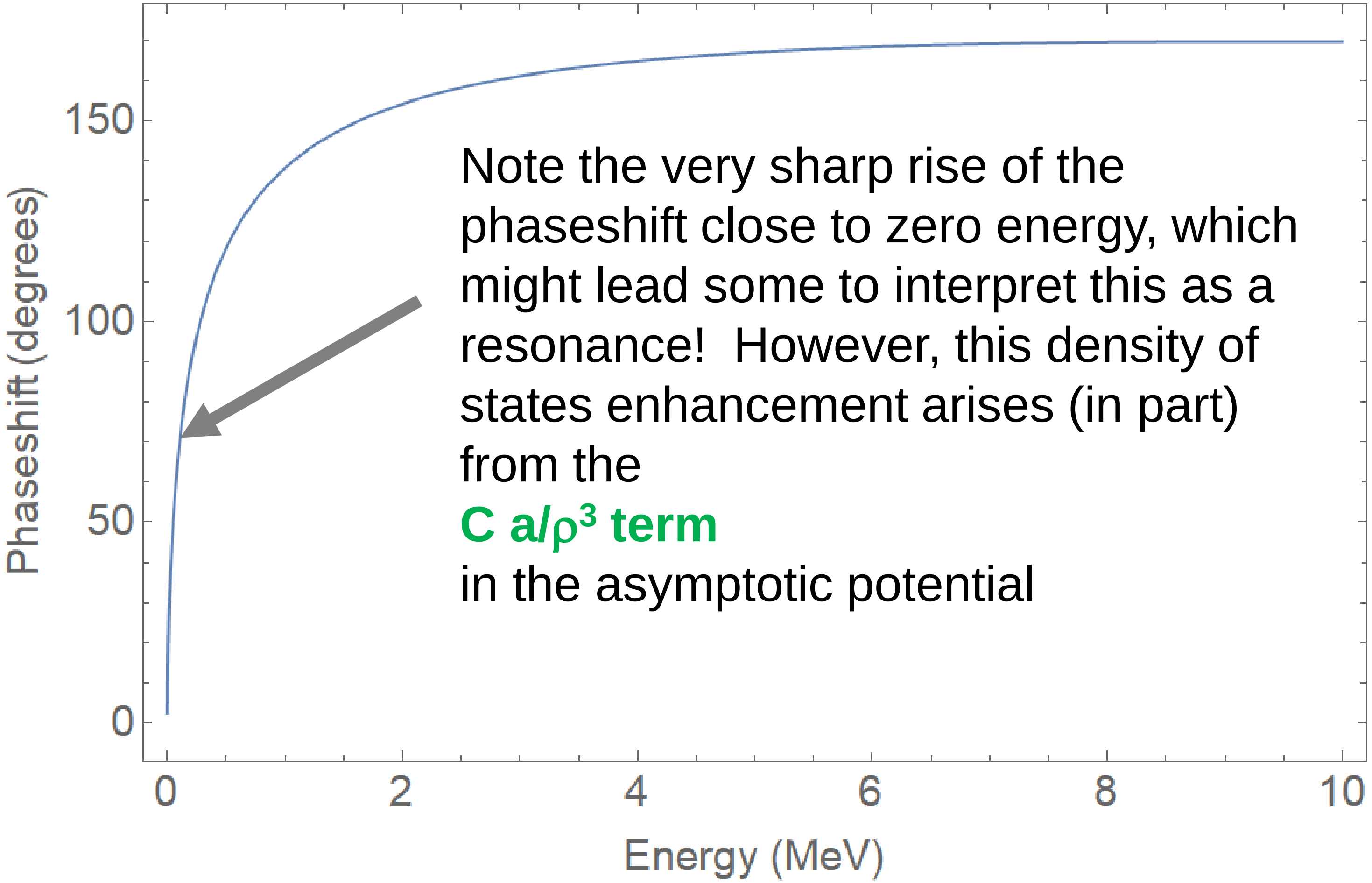
a(nn)=-18.98 fm for the Argonne AV8' potential, similar for AV18.

The most attractive hyperspherical potential curves for the 4n and 3n systems, obtained using the AV8’ n-n interaction potentials (magenta)



The converged potentials are clearly totally repulsive, with <u>no sign of a local maximum</u> that can trap probability in a resonance.

Next consider the scattering phaseshift in the lowest 4n potential



Note the very sharp rise of the phaseshift close to zero energy, which might lead some to interpret this as a resonance! However, this density of states enhancement arises (in part) from the

C a/ρ^3 term

in the asymptotic potential

While this causes the phaseshift to vary as $E^{1/2}$ near $E \rightarrow 0$ and the D.O.S. to diverge there, it reflects physics at <u>very long range</u>, and seems <u>unlikely</u> to cause the extra 3n or 4n events near E=0. I will elaborate further on this point below.

A key point: The lowest energy behavior is controlled by the longest range portions of the potential curve

Recall what we learned from Efimov physics, treated in the adiabatic hyperspherical framework (e.g. Zhen & Macek, 1988 Phys. Rev. A):

For three identical bosons, with finite particle-particle scattering length *a*, the asymptotic hyperradial potential was shown to behave for large |a| as

$$u_0(\rho) \to \frac{\hbar^2}{2\mu}\left(\frac{l_{\rm eff}(l_{\rm eff}+1)}{\rho^2} + C\frac{a}{\rho^3}\right)$$

Noninteracting term, For 3 identical bosons $l_{\rm eff} = \frac{3}{2}$

$$\left\langle \sum_{i<j} \frac{2\pi a\hbar^2}{m_{red}}\delta(\vec{r}_i - \vec{r}_j)\right\rangle = C\frac{a\hbar^2}{2\mu\rho^3}$$

But in the UNITARITY LIMIT, *a→infinity*, as Efimov taught us, the potential changes to the following form:

$$u_0^{univ}(\rho) = \frac{l_{\rm eff,u}(l_{\rm eff,u}+1)\hbar^2}{2\mu\rho^2}$$

$$l_{\rm eff,u}(l_{\rm eff,u}+1) \to -s_0^2 - \frac{1}{4}$$

Here are the values of these parameters for the 4n and 3n systems:

N	$(LS)J^\pi$	l_{eff}	C	$l_{\mathrm{eff,u}}^{(a)}$ Our numerical result	$l_{\mathrm{eff,u}}^{(b)}$ Yin and Blume, 2015 PRA
3	$(1\frac{1}{2})\frac{3}{2}^-$	$5/2$	15.22	1.275	1.2727(1)
4	$(00)0^+$	5	86.68	2.027	2.0091(4)

$$u_0(\rho) \rightarrow \frac{\hbar^2}{2\mu}\left(\frac{l_{\mathrm{eff}}(l_{\mathrm{eff}}+1)}{\rho^2} + C\frac{a}{\rho^3}\right)$$

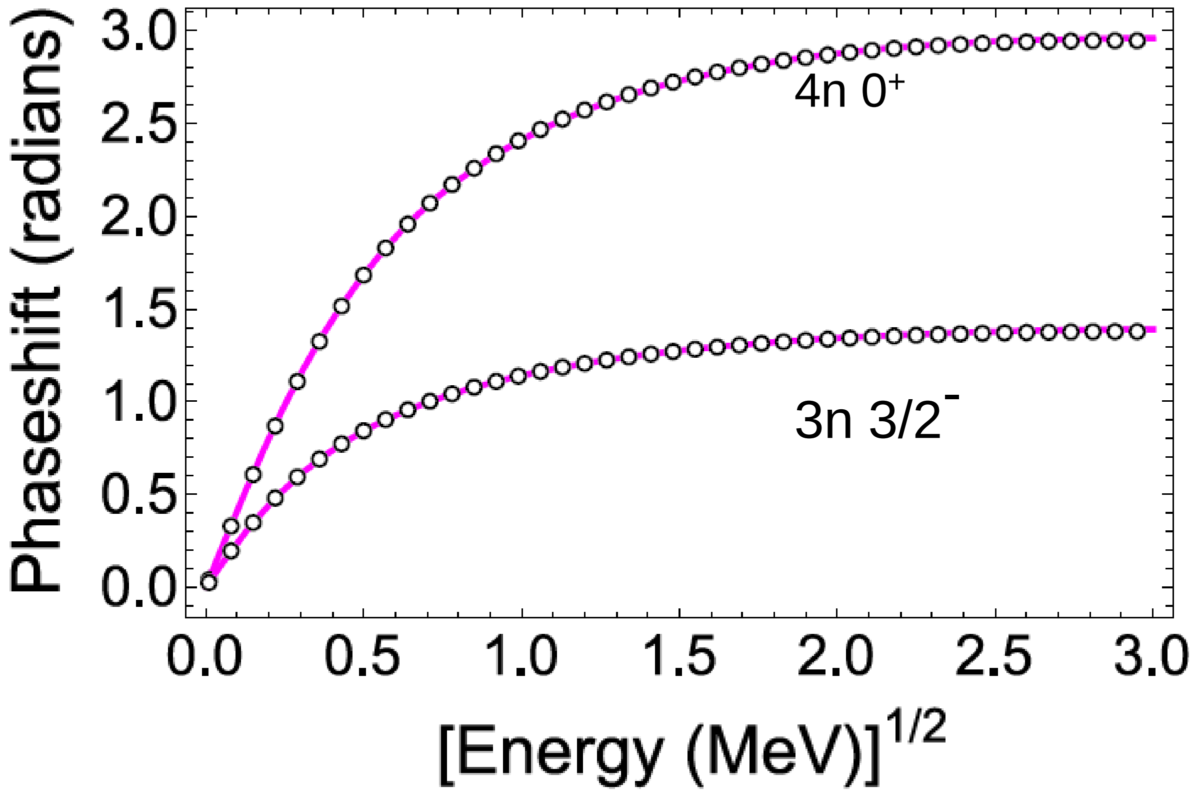
$$u_0^{univ}(\rho) = \frac{l_{\mathrm{eff,u}}(l_{\mathrm{eff,u}}+1)\hbar^2}{2\mu\rho^2}$$

Implications of the attractive a/ρ^3 potential at long range

One can readily derive (e.g. using the Born approximation) that the limiting low energy phaseshift in such a potential of the form,

$$u_0(\rho) \rightarrow \frac{\hbar^2}{2\mu}\left(\frac{l_{\text{eff}}(l_{\text{eff}}+1)}{\rho^2} + C\frac{a}{\rho^3}\right)$$

is: $\delta \underset{k \rightarrow 0}{\rightarrow} -Cak/(2l_{\text{eff}} + 2l_{\text{eff}}{}^2)$ ← However, this form holds only out to about E ~ 0.2 – 0.3 MeV



Next, consider further implications of the low energy phaseshift behavior from the perspective of a Wigner-Smith time delay analysis.

$$Q(E) = i\hbar S dS^{\dagger}/dE$$

$$\rightarrow 2\hbar d\delta(E)/dE$$ (in the single-channel limit)

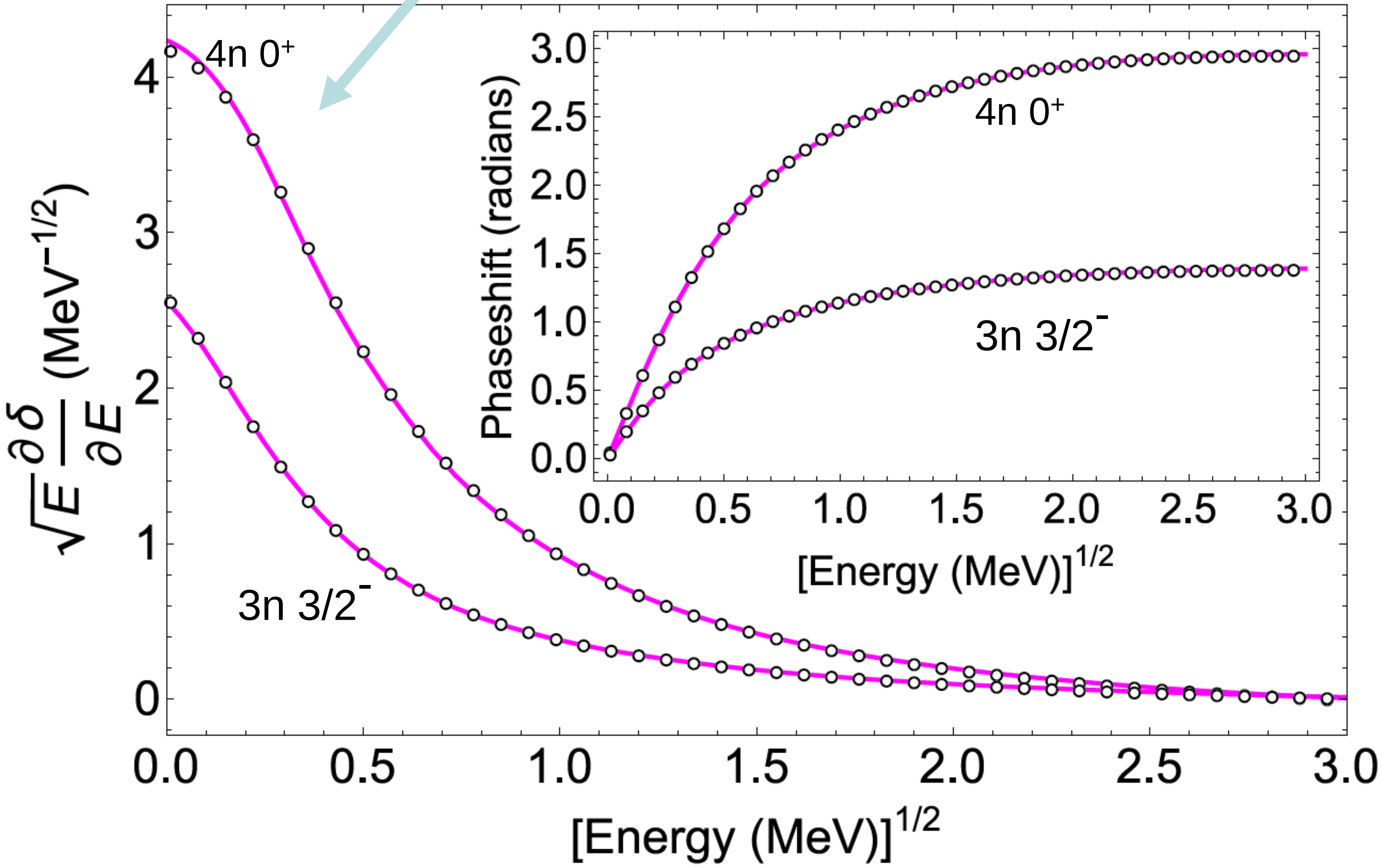
Moreover, Q(E) divided by $2\pi\hbar$ is the density of states

We can conclude that both the 3n and 4n systems have a divergent density of states proportional to 1/sqrt(E) at E→0, because the phaseshift is proportional to sqrt(E).

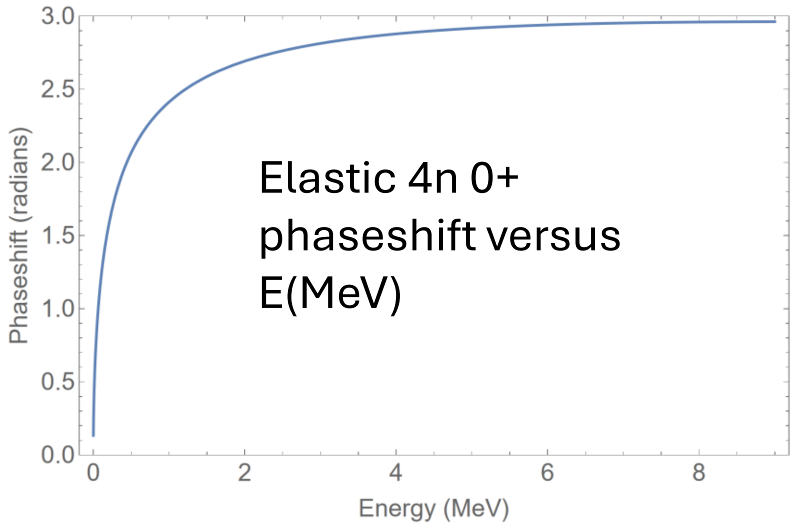
However, this comes from very long range physics, very likely not controlling the production of the 3n or 4n state

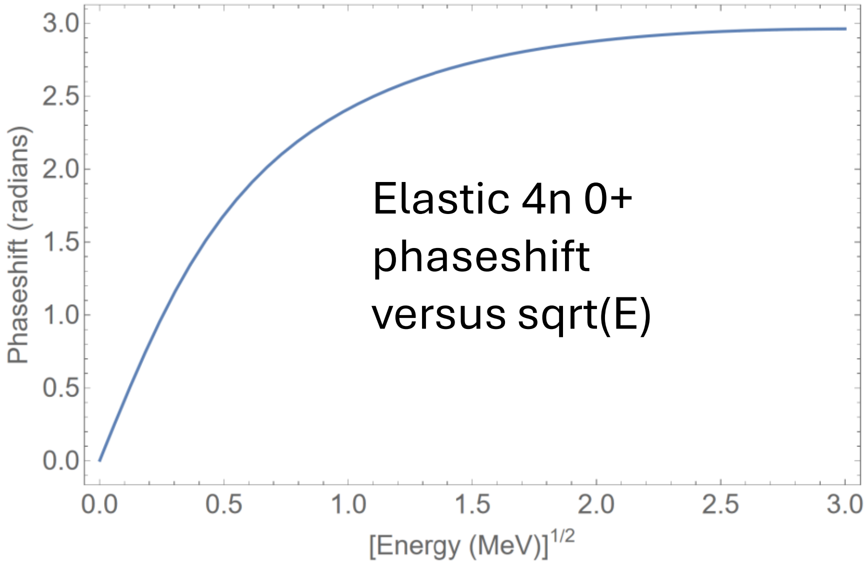
E. P. Wigner, Lower limit for the energy derivative of the scattering phase shift, Phys. Rev. 98, 145 (1955).
F. T. Smith, Lifetime matrix in collision theory, Phys.Rev. 118, 349 (1960).

Rescaled time delays for the 3n and 4n systems, showing that they are finite at E→0, but only when multiplied by sqrt(E)

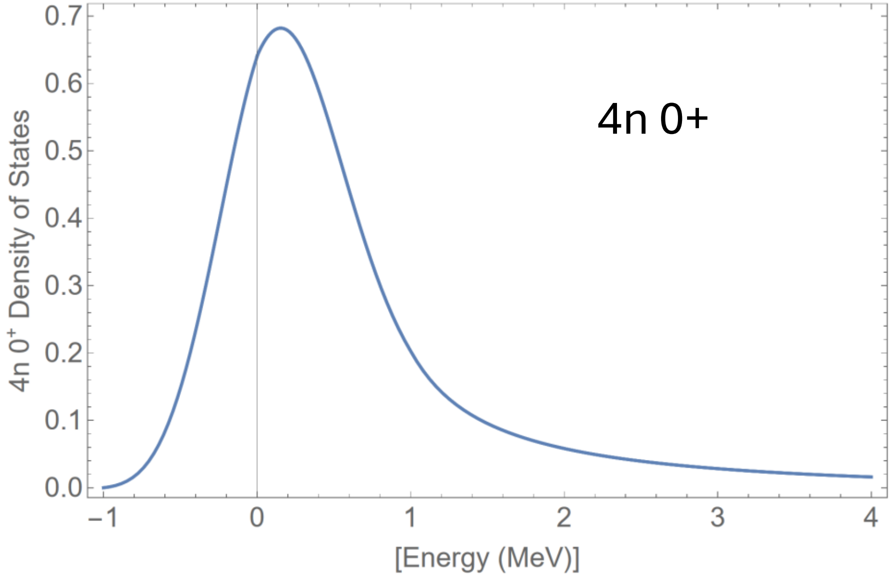


Excellent threshold laws reference → J. Phys. B: At. Mol. Opt. Phys. **33** (2000) R93–R140
H R Sadeghpour, J L Bohn, M J Cavagnero, B D Esry, I I Fabrikant, J H Macek and A R P Rau





Long range density of states peaks obtained after convolving with 1 MeV resolution function



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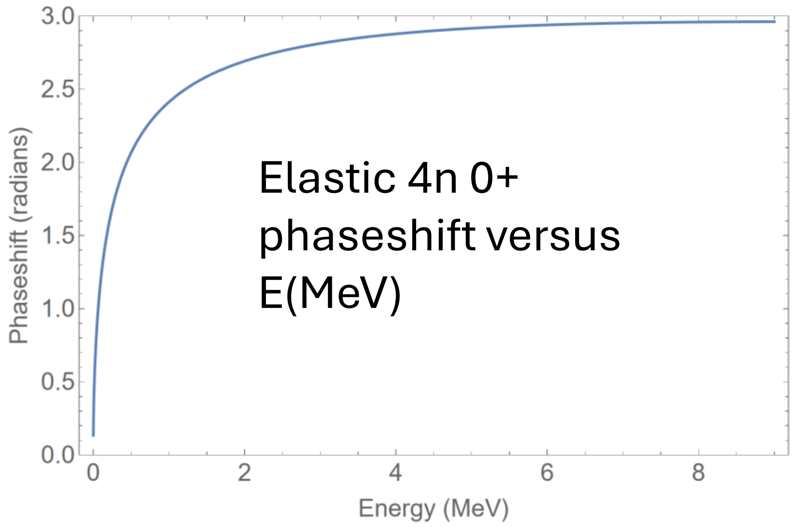
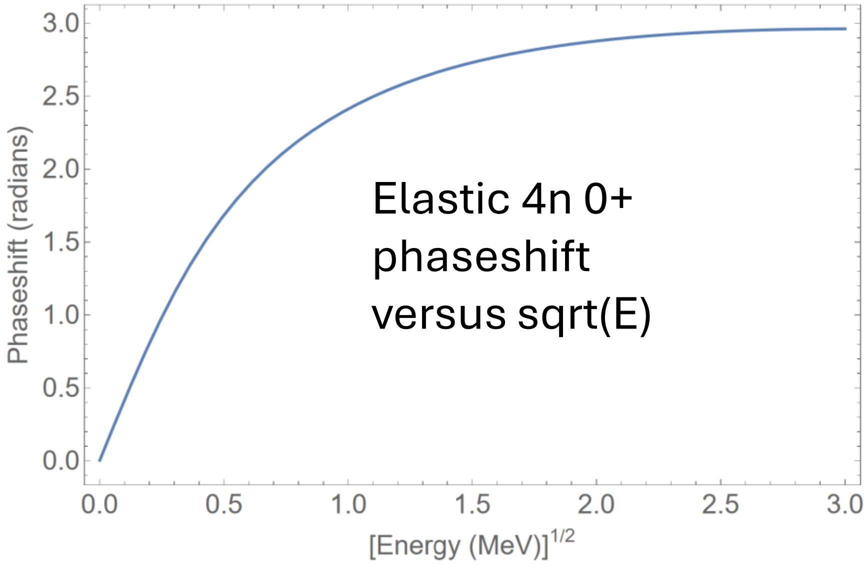
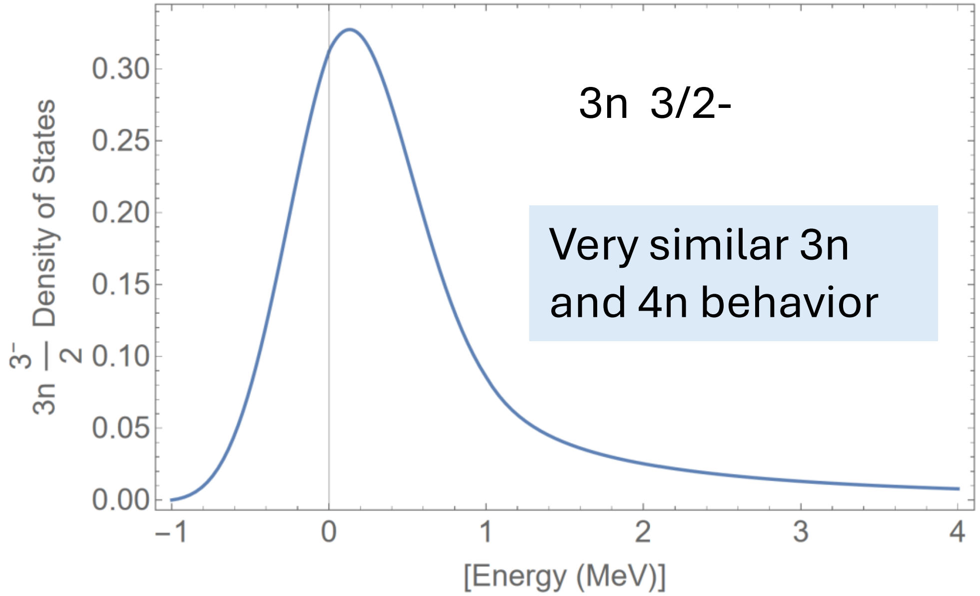
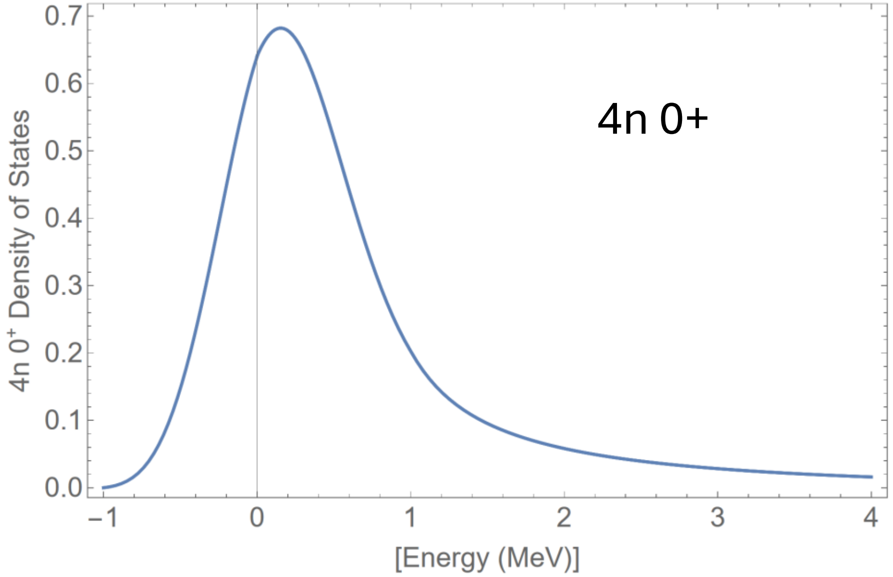
H R Sadeghpour, J L Bohn, M J Cavagnero, B D Esry, I I Fabrikant, J H Macek and A R P Rau





Long range density of states peaks obtained after convolving with 1 MeV resolution function

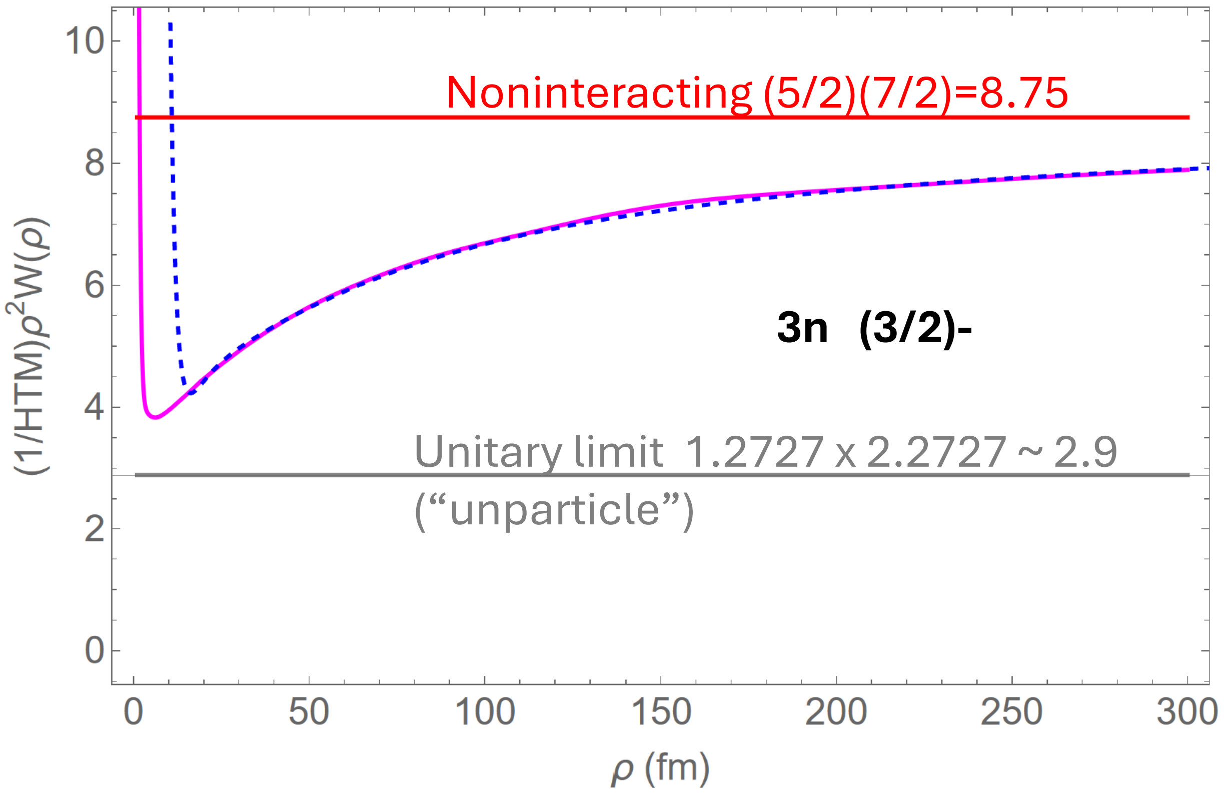




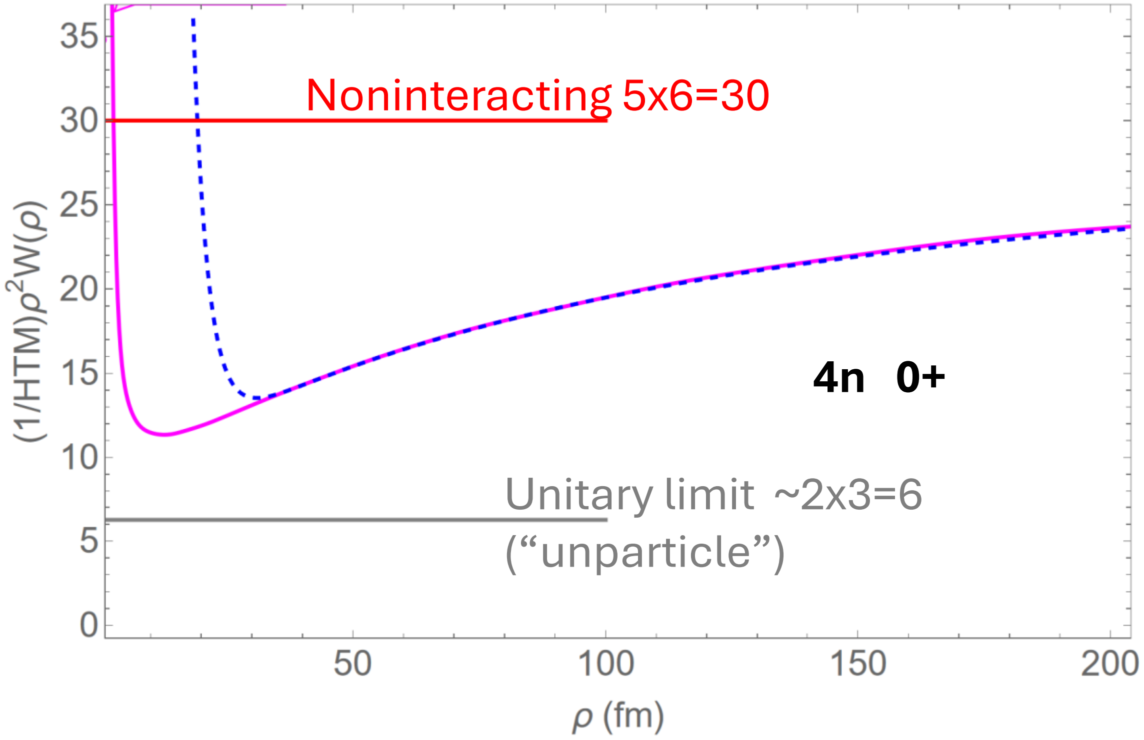
Fit to the asymptotic potentials, multiplied by $2m\,\rho^2/hbar^2$

This is a study of the 3n,4n long range attractions, showing a very slow convergence of the potentials

$8.75 + b/\rho + c/\rho^2 + d/\rho^3 + f/\rho^4$ is the blue dashed fit below



$30 + b/\rho + c/\rho^2 + d/\rho^3 + f/\rho^4$ is the blue dashed fit



These blowups of the long range rescaled potentials show that the very long range attraction is what causes the phaseshift rise (nonresonantly)

Next, let’s investigate the energy dependent probability of the 3n and 4n systems to reach small hyperradii, which gives the energy dependence relevant to any matrix element of an operator that produces 3n or 4n via a short-range process

I will now argue that this analysis can be carried out accurately using WKB to calculate the tunneling under the centrifugal barrier down to short distances, as a generalization of the Wigner threshold law

Aside on the interpretation of Wigner threshold law factors as a tunneling integral

Here is a brief description about how one can understand the Wigner threshold law for any transition matrix element that is controlled by the short-range part of the near-threshold wavefunction. Consider a general potential with the following asymptotic form:

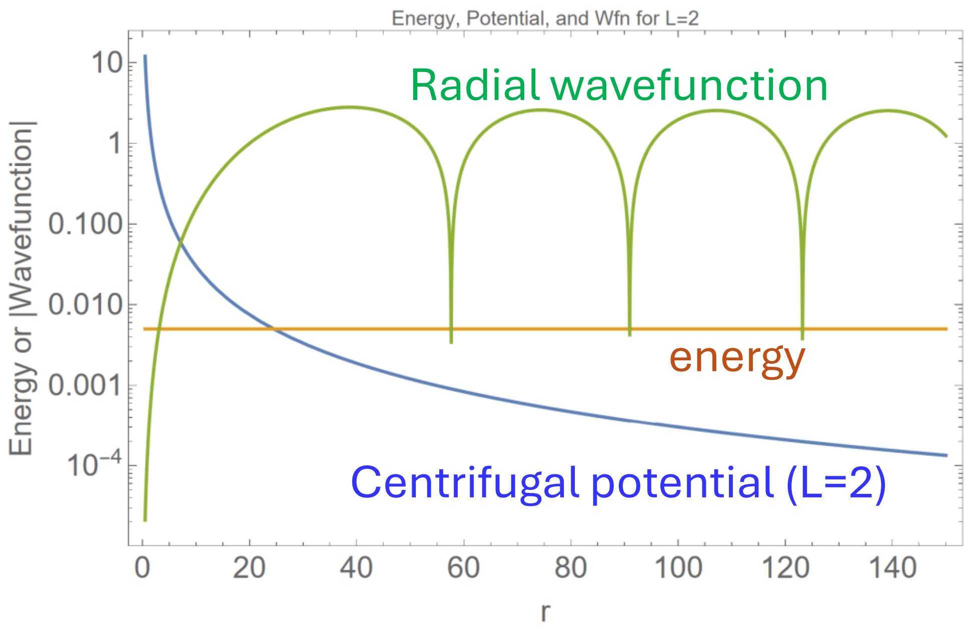
$$V(\rho) \to \frac{\ell(\ell+1)\hbar^2}{2\mu\rho^2}, \text{as} \quad \rho \to \infty$$

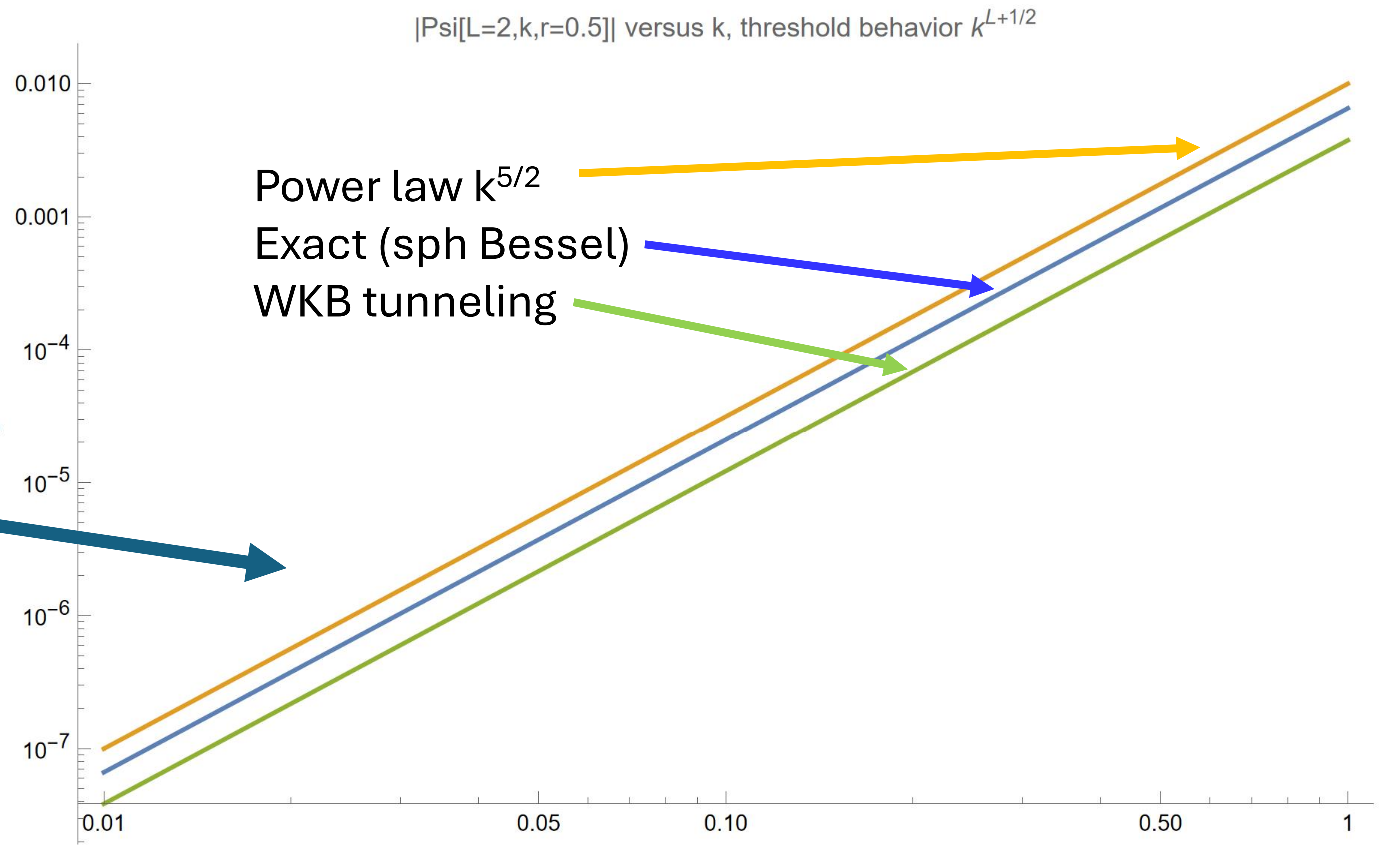
It is not often taught in textbooks, but the Wigner threshold law energy-dependent factor in the transition matrix element can be viewed as the tunneling amplitude through the centrifugal barrier. To see this, consider the WKB tunneling integral from the long range turning point at $\rho \approx \frac{\ell+\frac{1}{2}}{k}$ into a short distance:

$$\psi(k, \rho_{\text{small}}) \propto \exp\left(-\int_{\rho_{\text{small}}}^{\frac{\ell+\frac{1}{2}}{k}} Re\sqrt{\frac{(\ell+\frac{1}{2})^2}{\rho'^2} - k^2 d\rho'}\right)$$

$$\approx \exp\left(-(\ell+\frac{1}{2})ln(k\rho_{\text{small}})\right) \propto k^{\ell+\frac{1}{2}}$$

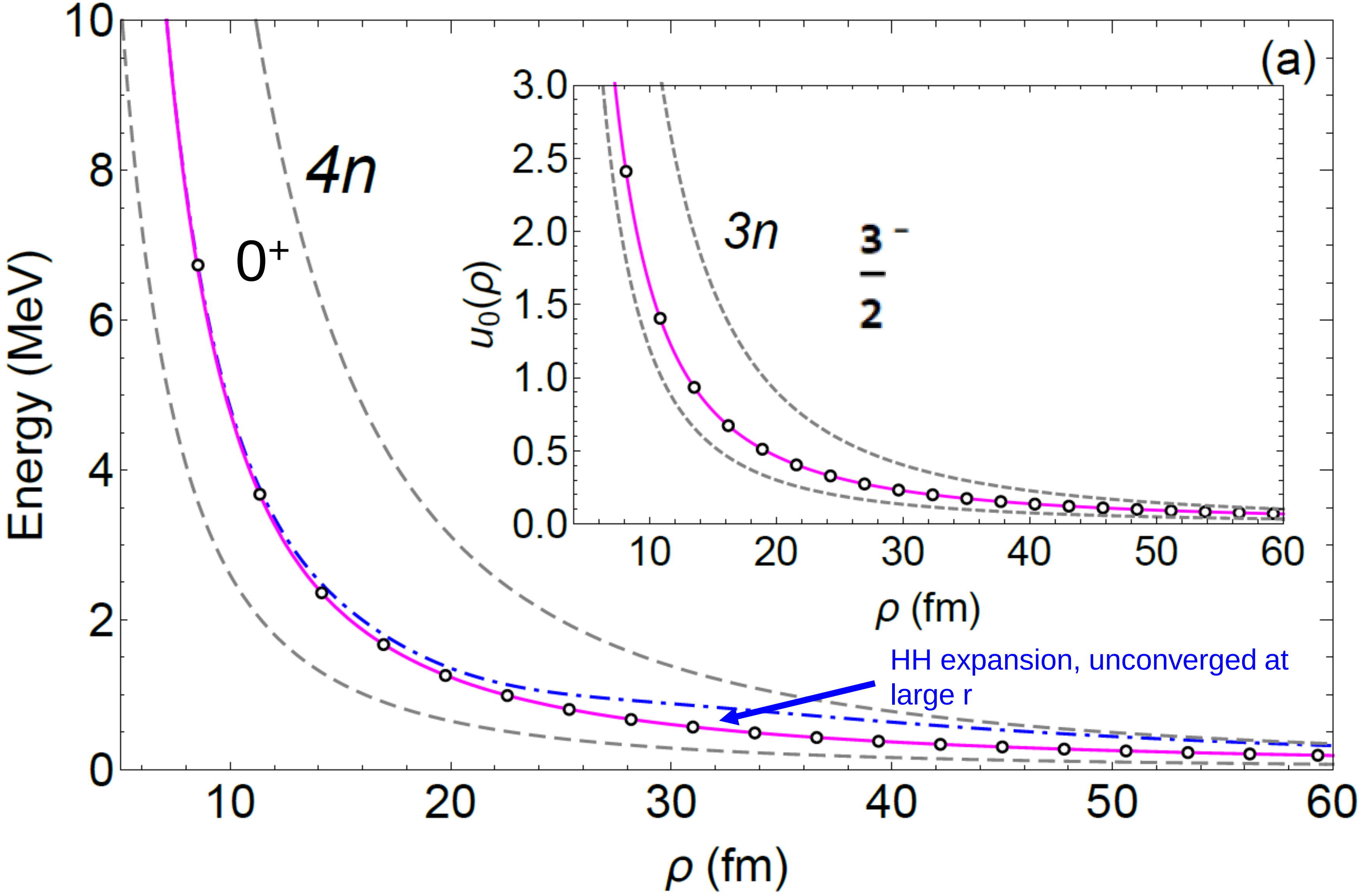
Demonstration that this WKB argument gives the correct Wigner threshold law for an L=2 wavefunction





For this d-wave example, we confirm that both the exact spherical Bessel solution AND the WKB tunneling calculation agree on the Wigner threshold law energy dependence at low energy, namely Psi ~ $k^{(L+1/2)}$

The most attractive hyperspherical potential curves for the 4n and 3n systems, obtained using the AV8' n-n interaction potentials (magenta)



The converged potentials are clearly totally repulsive, with <u>no sign of a local maximum</u> that can trap probability in a resonance.

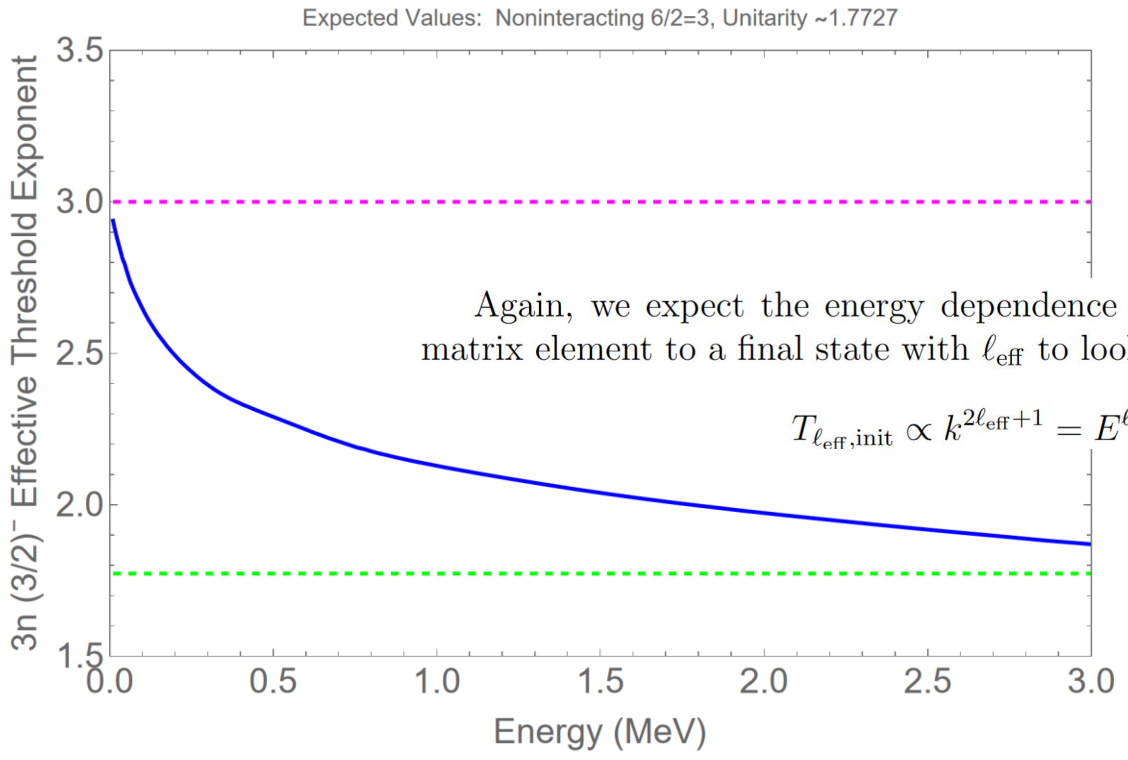
Checking the expected threshold behavior of a transition matrix element involving the **3-neutron 1⁻ final state wavefunction** as a function of the relative energy

Note that if the final state wavefunction at short range is controlled by the outermost classical turning point at $R \approx \frac{\ell_{\rm eff}+1/2}{k}$ and if that turning point varies slowly with k, then we can define an effective energy-dependent threshold law exponent γ in $\psi \propto k^\gamma$, using the formula:

$$\gamma(k) \approx k \frac{d}{dk} \ln\left(\psi(k, \rho_{\rm small})\right)$$

Analysis is based on the following tunneling integral with our hyperspherical potential curves

$$\psi(k, \rho_{\rm small}) \propto \exp\left(-\int_{\rho_{\rm small}}^{\frac{\ell+\frac{1}{2}}{k}} Re\sqrt{\frac{2\mu U(\rho)}{\hbar^2} + \frac{(\frac{1}{2})^2}{\rho'^2} - k^2 d\rho'}\right)$$



←**expected threshold exponent 3 very close to threshold (non-interacting exponent)**

Again, we expect the energy dependence of any short range transition matrix element to a final state with $\ell_{\rm eff}$ to look like:

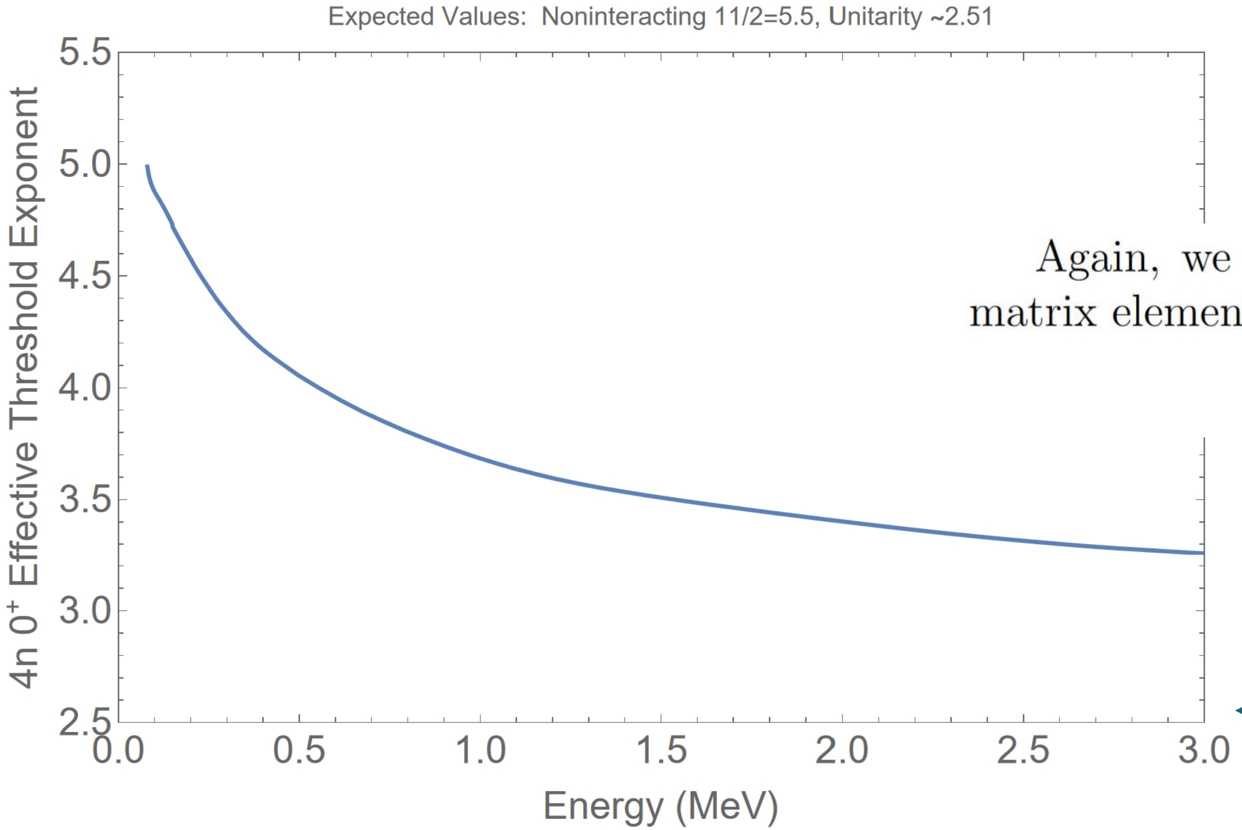
$$T_{\ell_{\rm eff},{\rm init}} \propto k^{2\ell_{\rm eff}+1} = E^{\ell_{\rm eff}+1/2}$$

expected unitary exponent ~1.77 very close to threshold (unitarity limit, Hammer & Son PNAS suggest that ~1.77 should be the observable threshold exponent for the 3n system in 1- symmetry)

Checking the expected threshold behavior of a transition matrix element involving the **4-neutron 0+ final state wavefunction** as a function of the relative energy

Note that if the final state wavefunction at short range is controlled by the outermost classical turning point at $R \approx \frac{\ell_{\text{eff}}+1/2}{k}$ and if that turning point varies slowly with k, then we can define an effective energy-dependent threshold law exponent γ in $\psi \propto k^{\gamma}$, using the formula:

$$\gamma(k) \approx k \frac{d}{dk} \ln\left(\psi(k, \rho_{\text{small}})\right)$$



←**expected threshold exponent 5.5 very close to threshold (non-interacting exponent)**

Again, we expect the energy dependence of any short range transition matrix element to a final state with ℓ_{eff} to look like:

$$T_{\ell_{\text{eff}},\text{init}} \propto k^{2\ell_{\text{eff}}+1} = E^{\ell_{\text{eff}}+1/2}$$

expected unitary exponent ~2.51 very close to threshold (unitarity limit, Hammer & Son PNAS state that ~2.5 to 2.6 should be the observable threshold exponent)

Summary of our 3n and 4n studies

Basically the theory prediction looks very similar for the 3n and 4n systems, based on our adiabatic hyperspherical quantitative study:

1. There is clearly no possibility of a 3n or 4n resonance, based on our current understanding of the neutron-neutron interaction potential

2. There is an enhancement of the density of states at low energy, for both the 3n and 4n systems, but this is associated with very long range physics, and it seems <u>unlikely to explain any excess low energy events</u> as are seen in the 4n experiments

3. Any short range matrix transition matrix element reaching the final state hyperradial potentials that we have calculated does obey the expected Wigner threshold law, but it holds only over a very small energy range, less than or about 0.1 MeV.

4. Our analysis of the energy dependent exponent of a transition matrix element (or short range wavefunction) shows that over 0.1 – 2 MeV, it varies with energy in the range between the non-interacting exponent and the unitary limiting exponent, which we can predict quantitatively for the symmetries we have studied. (And this too shows no resonant behavior for the 3n or 4n system.)

Pushing towards 5-particle adiabatic hyperspherical calculations

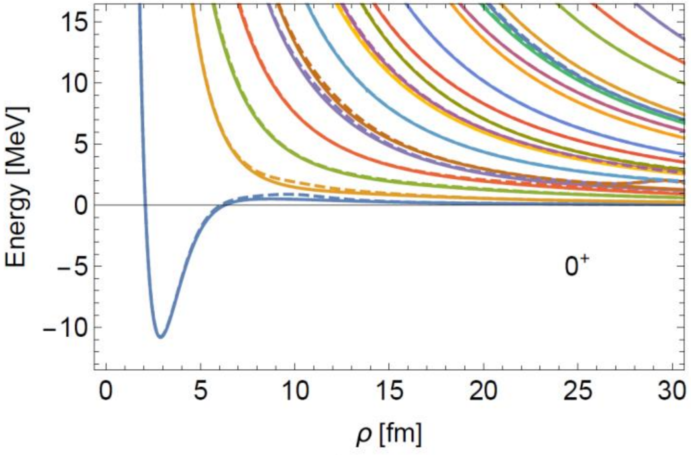
1. Alpha particle + 2, 3, or 4 neutrons (recent arXiv preprint)

2. The five boson recombination problem (unpublished and preliminary)

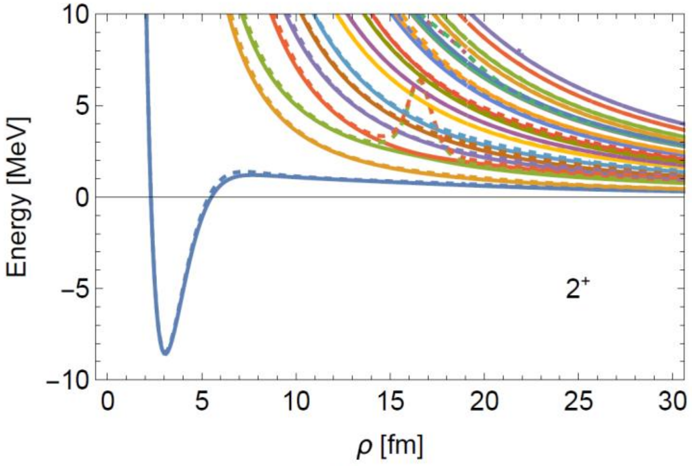
Moving to neutron rich helium isotopes, namely ^{6}He, ^{7}He, and ^{8}He

Preprint: see arXiv: 2407.17668, M. Higgins & CHG

^{6}He in two symmetries, showing generally good adiabaticity



(a)



(b)

^{8}He in 0^+ symmetry, showing stronger nonadiabatic channel coupling around ρ=10 fm

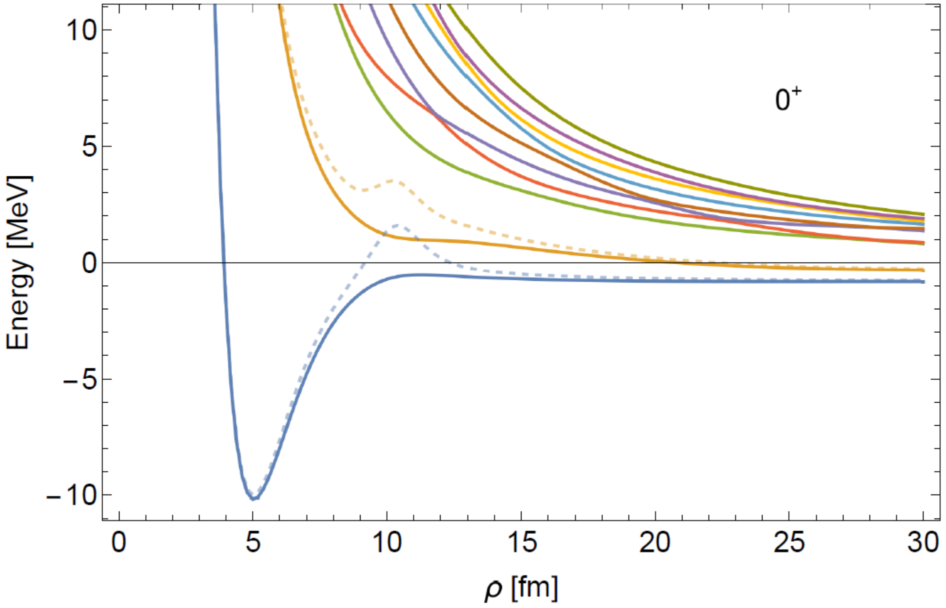
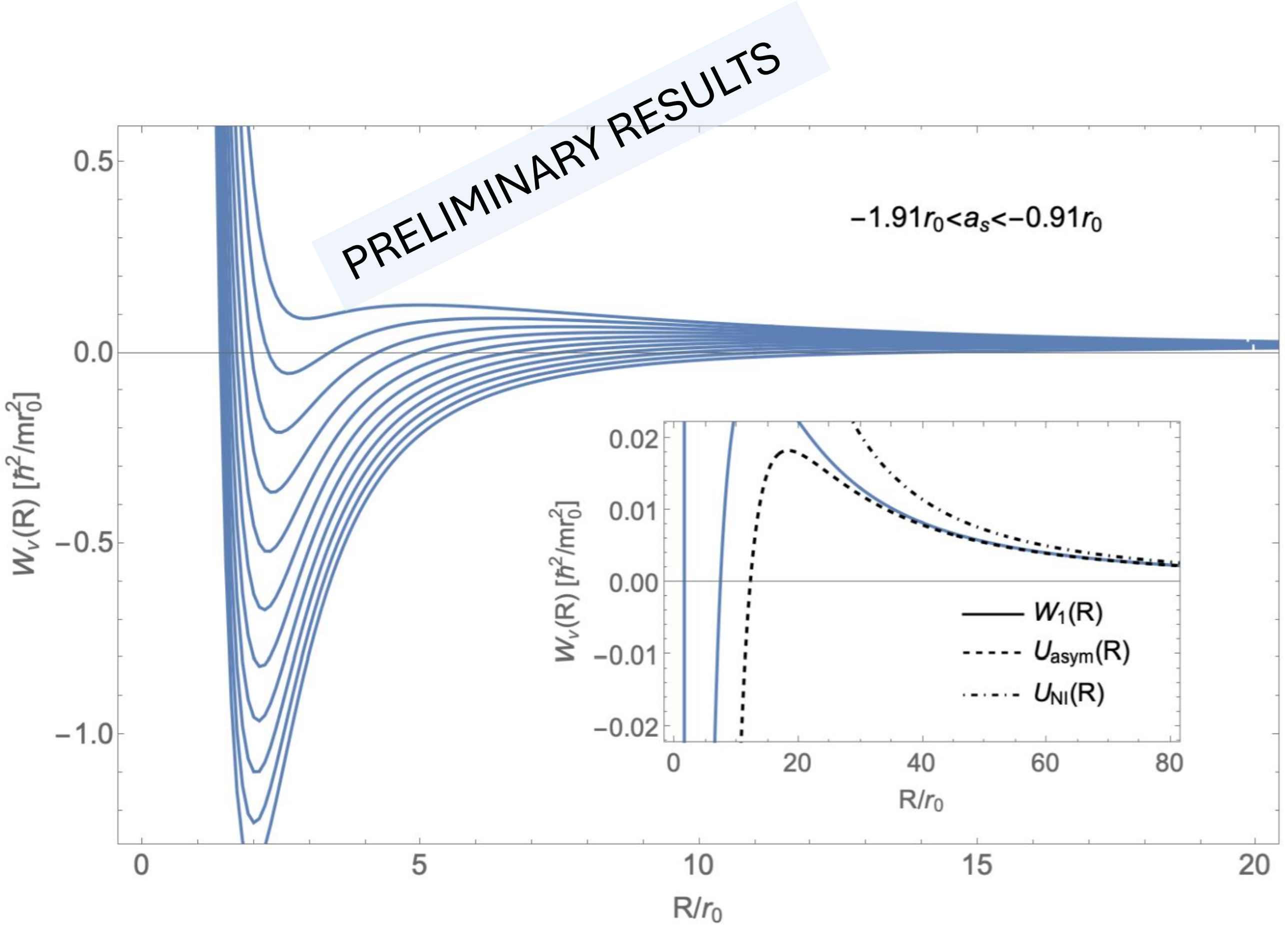


FIG. 9. The lowest few adiabatic potential curves for the ^{4}He + $4n$ system in the $(L^\pi, S)J^\pi = (0,0)0^+$ symmetry. The solid curves are the adiabatic potentials and the dashed curves are the effective potentials, which includes the diagonal second–derivative coupling term. Only the lowest two effective potentials are shown here.

Further advances in the five body problem using the adiabatic hyperspherical representation for bound state properties and collisional dynamics:

THE FIVE IDENTICAL BOSON PROBLEM



A+A+A+A+A ground state potential curve at many scattering lengths where there are no fewer than 5 or 6 particles that can bind

This is close to the region studied in the 5-body recombination experiment of Rudi Grimm's group, namely New Journal of Physics 15 (2013) 043040, Zenesini et al

How Efimov physics extends to more than 3 particles. This figure shows the schematic entrance channel **potential curve expected for N particles at negative 2-body scattering length**, From Mehta et al., 2009 PRL

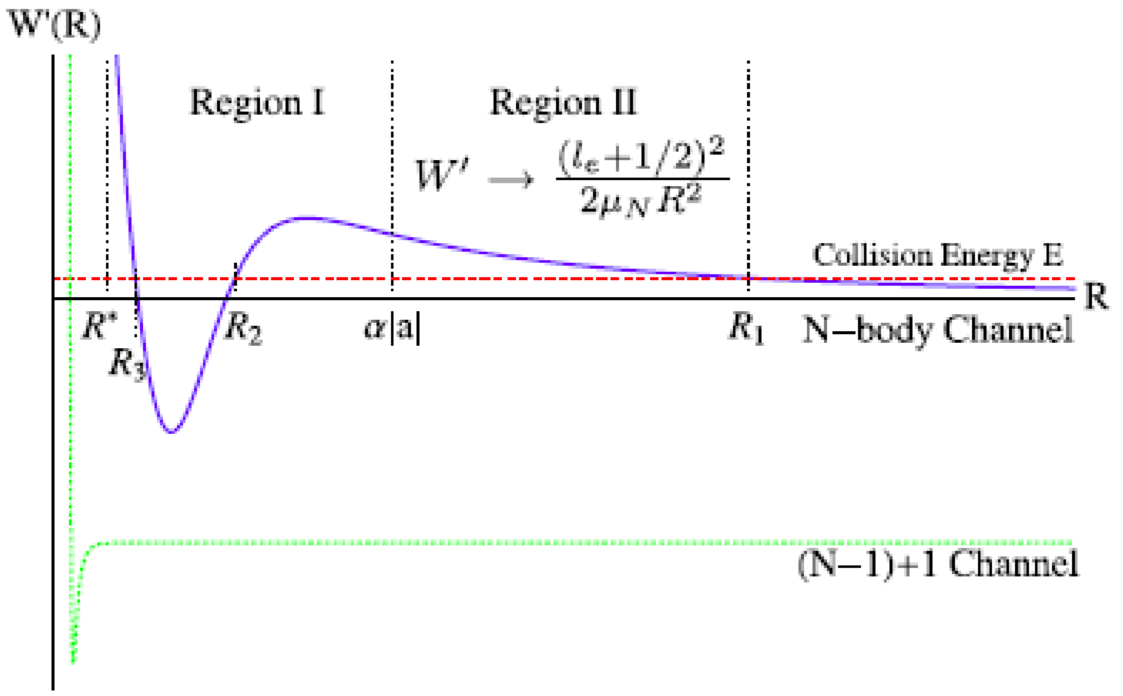


FIG. 1 (color online). A schematic representation of the N-boson hyperradial potential curves is shown. When a metastable N-boson state crosses the collision energy threshold at $E = 0$, N-body recombination into a lower channel with $N - 1$ atoms bound plus one free atom is resonantly enhanced.

But before we could actually calculate the rate of 4-body recombination in an ultracold gas, we had to develop some scattering theory:

PRL 103, 153201 (2009)

A general theoretical description of N-body recombination

N. P. Mehta,[1,2] Seth T. Rittenhouse,[1] J. P. D'Incao,[1] J. von Stecher,[1] and Chris H. Greene[1]

[1] *Department of Physics and JILA, University of Colorado, Boulder, CO 80309*

[2] *Grinnell College, Department of Physics, Grinnell, IA 50112**

(Dated: March 24, 2009)

We present a formula for the cross section and event rate constant describing recombination of N particles in terms of general S-matrix elements. Our result immediately yields the generalized Wigner threshold scaling for the recombination of N bosons. We find that four-boson recombination is resonantly enhanced by the presence of metastable states in the entrance channel. Hence, recombination into a trimer-atom channel could be an effective mechanism for the formation of Efimov trimers.

And here it is, THE FORMULA for N-body recombination, i.e. for the process:
A+A+A+….+..A → A_{N-1}+A or A_{N-2}+A+A +…etc.

$$K_N^{0^+} = \frac{2\pi\hbar}{\mu_N} N! \left(\frac{2\pi}{k}\right)^{(3N-5)} \frac{\Gamma\left((3N-3)/2\right)}{2\pi^{(3N-3)/2}} \left|S_{f0}^{0^+}\right|^2$$

J. Phys. B: At. Mol. Opt. Phys. 43 (2010) 101002 (5pp)   doi:10.1088/0953-4075/43/10/101002

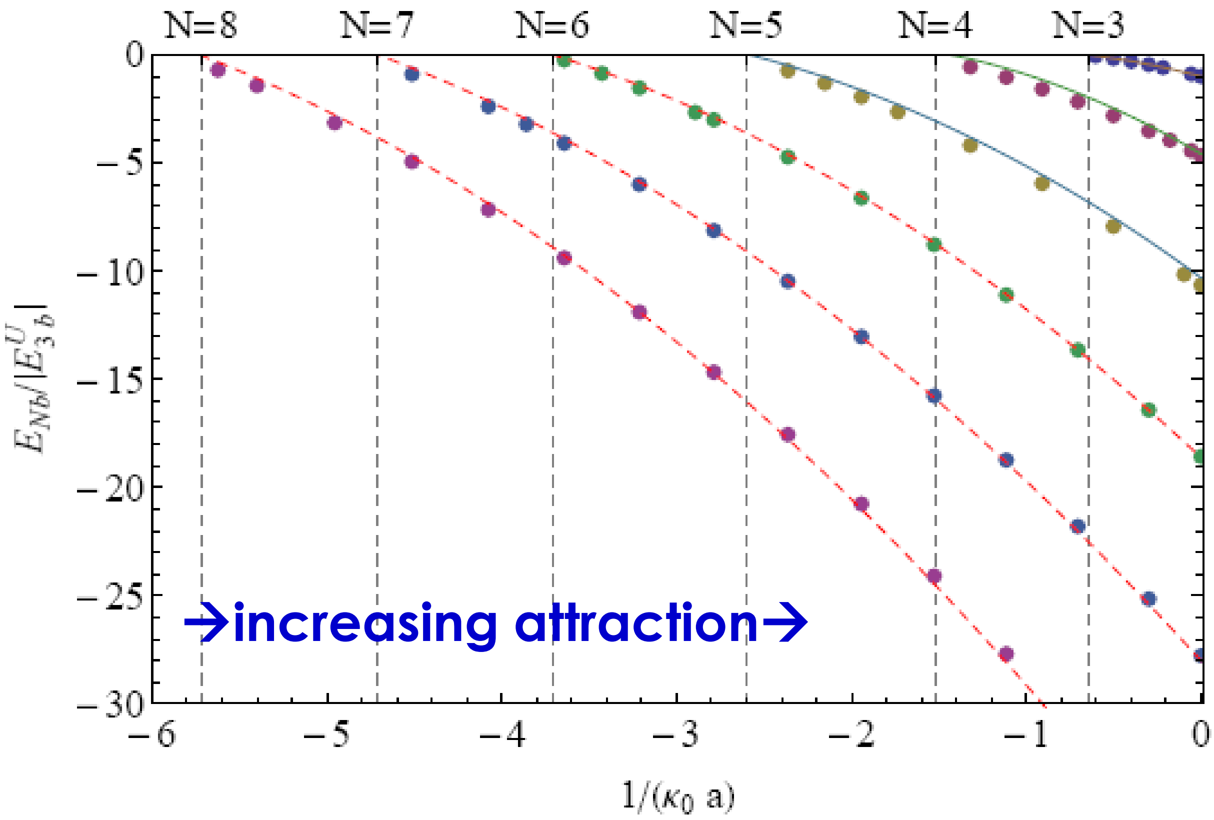
FAST TRACK COMMUNICATION

Weakly bound cluster states of Efimov character

MORE THAN 4 BOSONS: von Stecher's J. Phys. B article in 2010: combined study using correlated Gaussians, and diffusion Monte Carlo

Javier von Stecher

Clusters predicted up to N=13.



Javier von Stecher - and 2010 JPB and 2011 PRL

TABLE I: Energies at unitarity and scattering-length ratios that characterize weakly bound cluster states. The scattering length ratios can be transformed to an absolute scale using $1/(\kappa_0 a_{3b}) \approx 0.64$.

N	E_N^U/E_3^U	$a^*_{Nb}/a^*_{(N-1)b}$	N	E_N^U/E_3^U
4	4.66(4)	0.42(1)	9	49.9(6)
5	10.64(4)	0.60(1)	10	60.2(6)
6	18.59(5)	0.71(1)	11	70.1(7)
7	27.9(2)	0.78(1)	12	79.9(3)
8	38.9(3)	0.82(1)	13	88.0(7)

0.46(1) 0.65(2) 0.73(1) are latest revised/improved values from von Stecher,

Five- and Six-Body Resonances Tied to an Efimov Trimer

Javier von Stecher PRL **107**, 200402 (2011)

Remarkable prediction, that <u>**all**</u> larger cluster resonances are determined once the 3-body parameter is known!

How to tackle 5-body recombination for 5 free bosonic atoms with pairwise additive forces?

i.e. the reaction A+A+A+A+A→ A_3+A_2 or A_4+A or…

Start with the time-independent Schroedinger equation:

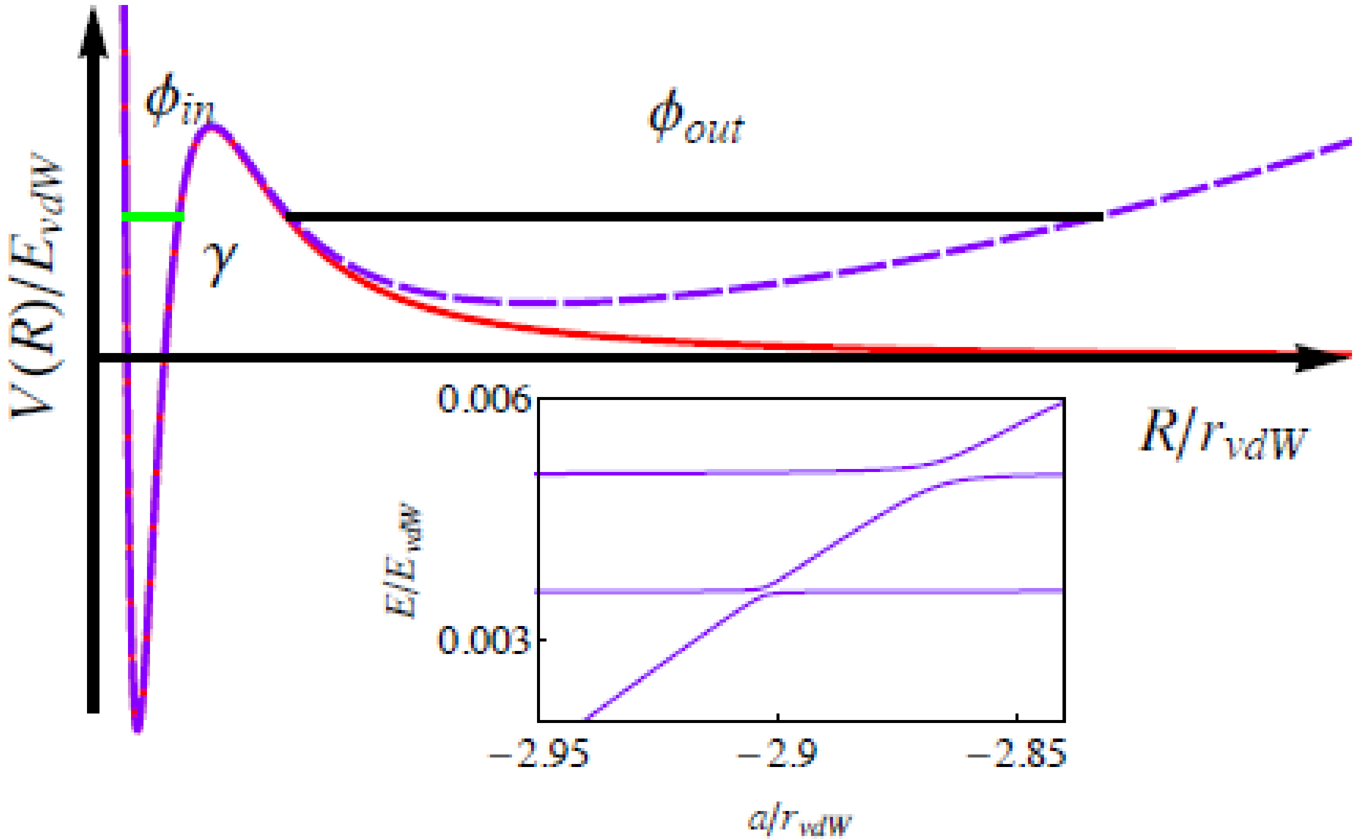
$$H= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + \frac{p_4^2}{2m_4} + \frac{p_5^2}{2m_5}$$

$$+ V(r_{12}) + V(r_{13}) + V(r_{14}) + V(r_{15}) + V(r_{23}) + V(r_{24}) + V(r_{25}) + V(r_{34}) + V(r_{35}) + V(r_{45})$$

After eliminating the center-of-mass degree of freedom, we're left with a 12-dimensional PDE to solve, which can be reduced to **a mere 9 dimensions** for J=0 states after going to the body frame.

Resonant five-body recombination in an ultracold gas of bosonic atoms

Mulliken-style potential energy versus hyperradius R for 5 free Cs atoms (solid red) or harmonically trapped (



NJP 2013

Alessandro Zenesini[1]‡, Bo Huang[1], Martin Berninger[1], Stefan Besler[1], Hanns-Christoph Nägerl[1], Francesca Ferlaino[1], Rudolf Grimm[1,2], Chris H Greene[3]§, Javier von Stecher[3,4]

Our article with the Innsbruck group: finally published by the New Journal of Physics, accepted for publication about 1 year later! Woohoo!

Resonant Five-Body Recombination in an Ultracold Gas

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Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria

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...ity of Colorado, Boulder, CO 80309, USA

...ry 10, 2012)

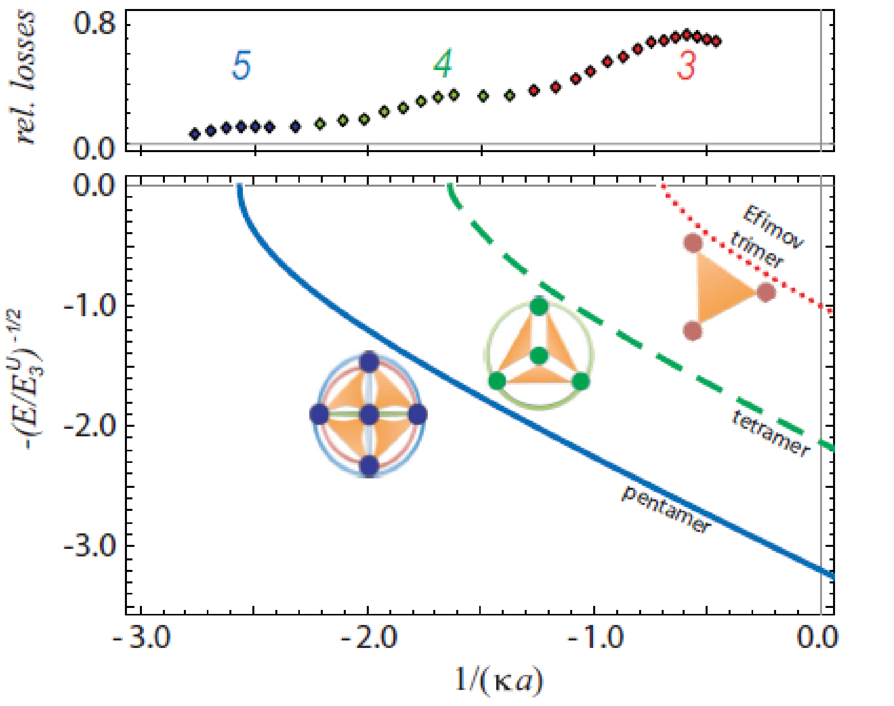
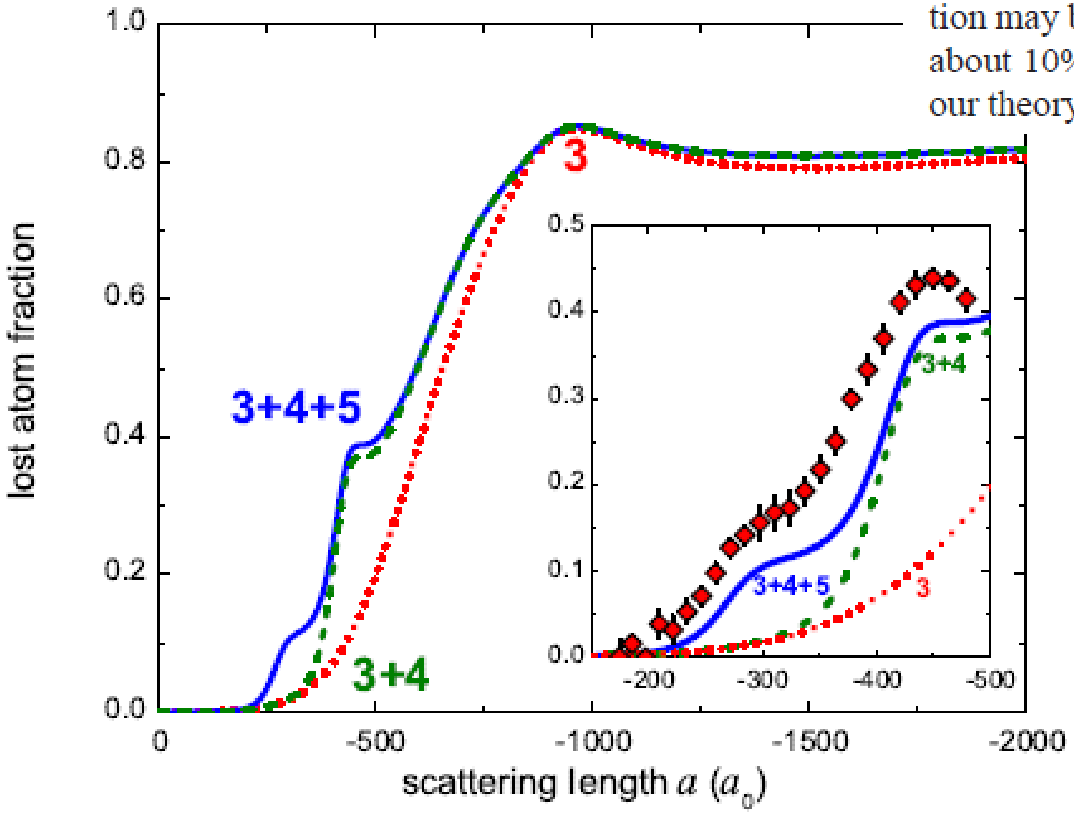


FIG. 1. (color online) N-body scenario in the region of negative two-body scattering length a. The lower panel shows the N-body binding energies as a function of the inverse scattering length. $E_3^U = (\hbar\kappa)^2/m$ is the trimer binding energy for resonant interaction. The dotted,

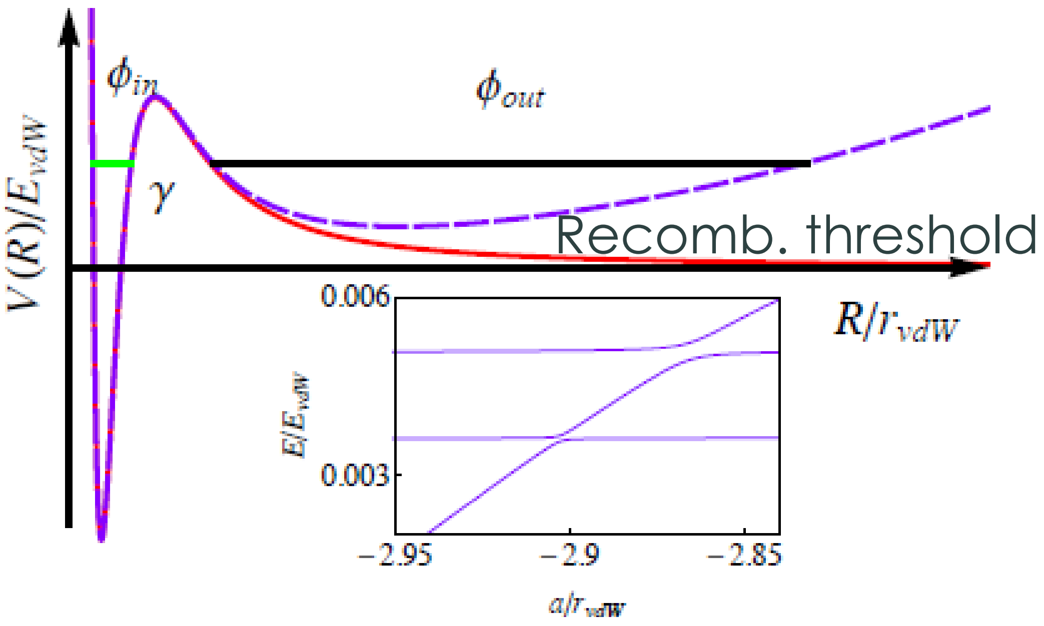
New Journal of Physics 15 (2013) 043040

count for the experimental observations. Remarkably, the resonance position $a_{5,-} = 0.64(2)\,a_{4,-}$ is in agreement with the theoretical predictions $0.65(1)\,a_{4,-}$ [37, 38]. However, quantitatively, the experimental values for L_5 are about 15 times larger than the calculated ones. To account for this, we introduce a corresponding scaling factor. We find that this deviation may be explained by a small error in the WKB phase γ of about 10%, which remains in a realistic uncertainty range of our theory.



NJP 15 (2013) 043040,
Zenesini et al.

FIG. 4. (color online) Calculated and measured fraction of loss atoms from an atomic sample of initially 5×10^4 atoms at a temperature of 80 nK after a hold time of 100 ms. The red dotted line corresponds to the losses predicted for three-body recombination only, while the dashed green line and the blue solid line include also contributions from four- and five body recombination, as quantified in this work. A



Schematic qualitative hyperspherical potential curve for 5 bosons at negative scattering length, "Mulliken style"

NJP 15 (2013) 043040, Zenesini et al.

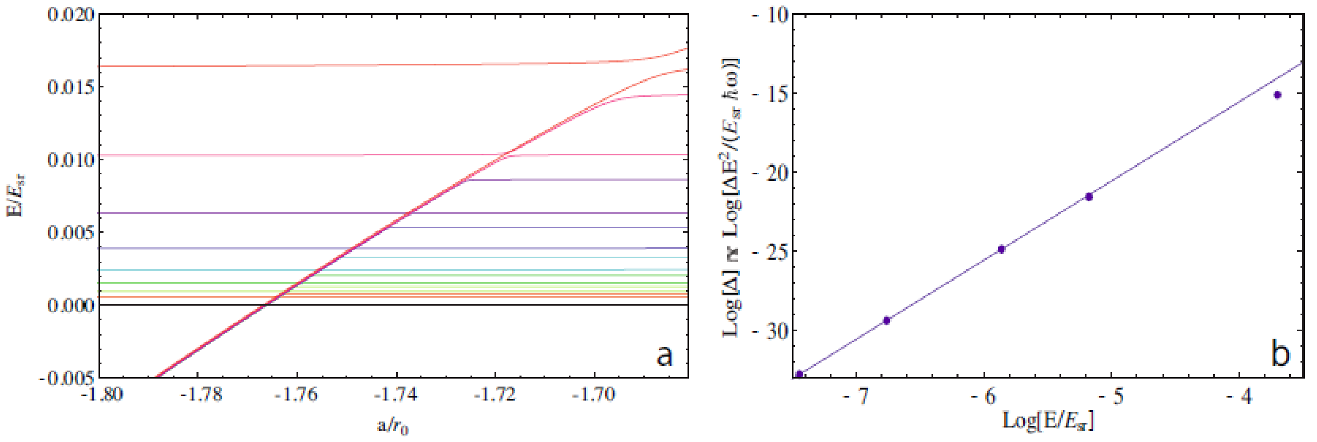


FIG. S1: (color online) (a) The lowest two eigenenergies of a trapped five body system are shown as functions of the scattering length for different trapping frequencies. Different colors represent different trapping frequencies. The combination of these states essentially describes the energy of the five-body state in the inner region of the potential $E_{mol}(a)$ (the diagonal curve). Here $E_{sr} = \hbar^2/(mr_0^2)$ and r_0 is the characteristic range of the two-body model potential that can be tuned to obtain the five-body resonance (i.e. $r_0 \sim 1.7 r_{vdw}$ where r_{vdw} is the van der Waals length). (b) The near-threshold behavior of Δ. The fitting of the lowest energy points leads implies that $\Delta \propto AE^b$. The lowest three points lead to $b \approx 5.004$ as expected from the known threshold behavior [4].

4-body recomb only

5-body recomb only

← **Separating the different N-body contributions**

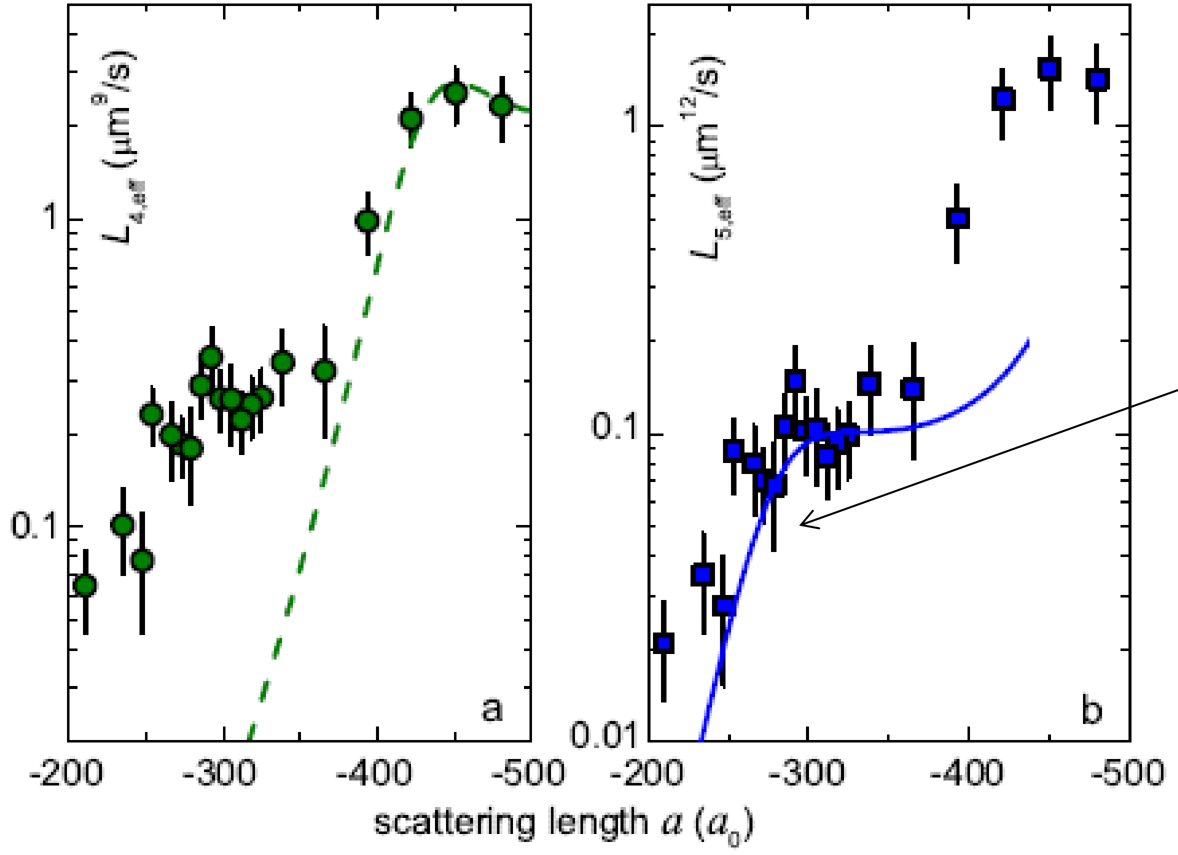


FIG. 3. (color online) Effective four- (a) and five-body recombination rates (b). The green dashed curve and the blue solid line follow the theoretical model for L_4 and L_5, respectively, with additional scaling factor for L_5; see text. The error bars include the statistical uncertainties from the fitting routine, the temperature and the trap frequencies.

Position of the predicted 4-body resonance and the 5-body resonance is in agreement with experiment! Kewl!

NJP 15 (2013) 043040, Zenesini et al.

A combined theoretical and experimental study of 5-body recombination

Conclusion:

Progress is underway to extend the predictive and analysis power of the adiabatic hyperspherical representation to handle more particles and more complex few-particle scenarios in both nuclear and atomic systems.

Thanks for listening!